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Well-covered triangulations: Part IV

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ABSTRACT

A graph G is said to be *well-covered* if every maximal independent set of vertices has the same cardinality. A plane (simple) graph in which each face is a triangle is called a (plane) *triangulation*. In the first of a sequence of three papers the authors proved that there are no 5-connected plane well-covered triangulations. Clearly, the only plane triangulation which is exactly 2-connected is the triangle K_3. Two subsequent papers culminated in the proof that there are exactly four well-covered plane triangulations which are exactly 4-connected.

It is the aim of the present paper to complete the characterization of well-covered plane triangulations by characterizing the infinite family of those well-covered triangulations of the plane which are exactly 3-connected.

1. Introduction

In 1969, the fourth author first proposed the study of graphs in which each maximal independent set of vertices has the same size and suggested that the name *well-covered* be applied to them [15]. More recently, this family of graphs has been studied in the context of distributed computing systems. (Cf. [22].)

It is well-known that the problem of determining the size of an independent set of vertices is *NP*-complete for graphs in general (cf. Karp [13]). However, for certain interesting sub-families such as *claw-free* graphs, the problem becomes polynomially solvable (cf. Minty [14] and Sbihi [19]). Clearly, the independent set problem has a polynomial solution for the class of well-covered graphs, but how does one recognize this class? It was shown independently by Chvátal and Slater [6] and by Sankaranarayana and Stewart [18] that the recognition problem for well-covered graphs is co-*NP*-complete. The problem remains co-*NP*-complete for the subclass of $K_{1,4}$-free graphs (cf. [5]). For more comprehensive surveys of well-covered graphs, see Plummer [16] and more recently, Hartnell [12].

Certain sub-families of well-covered graphs, including those which are claw-free, have been shown to be polynomially recognizable. (Cf. [1,3,4,2,7,8,17,20,21].) In this paper we complete the characterization of yet another such class: well-covered plane triangulations.

Clearly, the only plane triangulation having connectivity 2 is a single triangle. All other plane triangulations must have vertex connectivity 3, 4 or 5. In three earlier papers [9–11] it was proved that there are no 5-connected plane well-covered triangulations and that there are exactly four 4-connected well-covered triangulations of the plane, namely the graphs labeled R_6, R_7, R_8 and R_{12} in Fig. 1.1.

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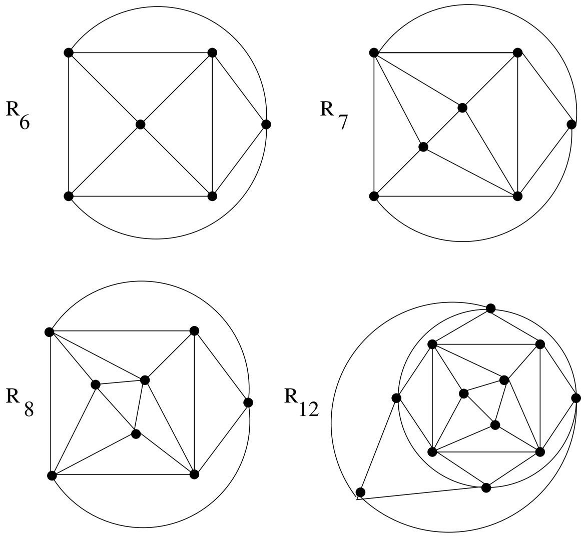

Fig. 1.1. The four 4-connected well-covered triangulations.

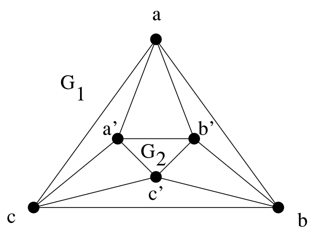

Fig. 2.1. The O-join.

In the present paper we complete the project of characterizing all plane well-covered triangulations by characterizing those which are exactly 3-connected. We call this class 3*CWCT*.

Take a finite planar graph where each component is a K_4 with at most one face non-empty, and arbitrarily add edges so as to form a connected planar graph where each face is a triangle. (See, for example, Fig. 5.1.) It is clear that each member of the resulting infinite family of plane triangulations, which we will call the K_4-*family* (denoted $\mathcal{K}$), is well-covered and exactly 3-connected. However, there are other 3-connected plane triangulations. We introduce a method of combining two plane triangulations to obtain a third by using an operation called an O-*join*. It is then shown that, using the members of $\mathcal{K}$, a small number of exceptional graphs and the process of O-joining in a manner to be described, one can obtain all members of 3*CWCT*.

Note that in this paper all graphs are finite and simple and if v is a vertex of a graph, $N[v]$ will denote the closed neighborhood of the vertex v; namely, $N[v] = N(v) \cup \{v\}$. The minimum degree of all vertices in G will be denoted by $\delta(G)$. Other terminology and notation will be introduced as needed.

2. Outline of proof

As this paper is long and rather complicated, we will begin by presenting an outline of our characterization. We begin with the graph family $\mathcal{K}$ and the exceptional graphs K_3, R_6, R_7, R_8 and R_{12}. Let $\mathcal{K} \cup \{K_3, R_6, R_7, R_8, R_{12}\}$ be denoted by $\mathcal{S}$.

Take two members of $\mathcal{S}$, say G_1 and G_2, and choose triangular faces *abca* in G_1 and $a'b'c'a'$ in G_2. Now connect these two triangles with six new edges aa', ab', bb', bc', cc', ca'. Then embed the resulting graph in the plane so that G_1 is exterior to triangle *abca* and G_2 is interior to triangle $a'b'c'a'$. (See Fig. 2.1.)

The resulting graph will be called an O-*join* of G_1 and G_2 at triangles *abca* and $a'b'c'a'$ respectively. This process can then be repeated with certain conditions prescribed for the two faces to be joined. These conditions will be detailed later in the paper. It is quite easy to see that if, subject to these conditions, two 3-connected well-covered plane triangulations are O-joined, the resulting graph is again 3-connected, well-covered and a triangulation. To prove the converse is more difficult. In fact, we will devote most of this paper to showing that if one has a 3-connected well-covered plane triangulation which is not a member of $\mathcal{S}$, that it must have come from smaller members of this family via a succession of O-joins.