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Safe set problem on graphs

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ABSTRACT

A non-empty subset S of the vertices of a connected graph $G = (V(G), E(G))$ is a *safe set* if, for every connected component C of $G[S]$ and every connected component D of $G - S$, we have $|C| \geq |D|$ whenever there exists an edge of G between C and D. If $G[S]$ is connected, then S is called *a connected safe set*. We discuss the minimum sizes of safe sets and connected safe sets in connected graphs.

1. Introduction

Facility location problems involve placing certain facilities on given topologies of living spaces (i.e. cities, buildings, etc.) so that given demands are efficiently served. These are intensively studied in the literature on combinatorial optimization (see [2]). We treat the following variation of the problem: for a given topology of a building, it is required to place temporary accident refuges in addition to business spaces like discussion or conference rooms. Each temporary refuge should be available for the staff in every adjacent business space; hence its capacity must be at least that of each adjacent business space. (To mitigate the space cost, we assume that each temporary refuge will be used by the people in at most one of the adjacent business space.) Subject to the topology of the building being given, how can the temporary refuges be efficiently located so that the amount of business spaces is maximized? To address this problem, we propose the following mathematical model, namely, a *safe set* of a graph.

We use [1] for terminology and notation not defined here. Only finite, simple graphs are considered. For a graph $G = (V(G), E(G))$, let $\delta(G)$ be the minimum degree of G, and $\alpha(G)$ be the independence number of G. We write $|G|$ as shorthand for $|V(G)|$. The subgraph of G induced by the subset $S \subseteq V(G)$ is denoted by $G[S]$. A connected component in $G[S]$ or $G - S$ will be called a component (so "connected" will be omitted). When A and B are vertex-disjoint subgraphs of G, the set of edges that join some vertex of A and some vertex of B is denoted by $E(A, B)$. Moreover, when $E(A, B) \neq \emptyset$, we say A is *adjacent* to B.

A non-empty subset $S \subseteq V(G)$ is a *safe set* if, for every component C of $G[S]$ and every component D of $G - S$, we have $|C| \geq |D|$ whenever $E(C, D) \neq \emptyset$. If $G[S]$ is connected, then S is called *a connected safe set*.

In our model, the graph G describes the topology of the building. The safe sets of G correspond to candidates for locations of the temporary refuges in the building.

As an initial step, let us observe some basic properties in safe sets.

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Proposition 1. *Let n be an integer with $n \geq 2$. Any connected graph G of order n has a connected safe set of size at most $\lceil n/2 \rceil$.*

Proof. Let T be a spanning tree of G, and S be any subset of $\lceil n/2 \rceil$ vertices such that $T[S]$ is connected. Clearly, S is a connected safe set. ■

The following two parameters are therefore well-defined. For a connected graph G, the *safe number* $s(G)$ of G is defined as $s(G) = \min\{|S| : S$ is a safe set of $G\}$, and the *connected safe number* $cs(G)$ of G is defined as $cs(G) = \min\{|S| : S$ is a connected safe set of $G\}$.

It is easy to see that the path on n vertices has $s(P_n) = cs(P_n) = \lceil n/3 \rceil$, and the cycle on n vertices has $s(C_n) = cs(C_n) = \lceil n/2 \rceil$.

Proposition 2. *Let G be a connected graph. Then $s(G) \leq cs(G) \leq 2s(G) - 1$.*

Proof. It is clear that $s(G) \leq cs(G)$. We prove the second inequality. Let S be a safe set of cardinality $s(G)$, and assume $G[S]$ is not connected.

Let H be the bipartite graph with bipartition (A, B), where A is the set of components of $G[S]$, B is the set of components of $G - S$, and there is an edge from $a \in A$ to $b \in B$ if there is an edge of G joining a vertex of a to a vertex of b.

The graph H is connected because G is connected. Since $G[S]$ is not connected, $|A| \geq 2$ and hence there exists a vertex of B with degree at least two. Let T be a spanning tree of H. Note that some vertex of B is not a leaf of T.

Let T' be the rooted tree obtained from T by deleting any leaf belonging to B, and choosing as root any remaining vertex $r \in B$. The tree T' has a matching in which every vertex in $b \in B \cap T'$ is paired with a child vertex in $a_b \in A$. By definition of a safe set, $|b| \leq |a_b|$. Consequently, $V(T')$ is a connected safe set.

Since r is not a leaf of T', there exists $a \in N_{T'}(r)$ which is not matched with any vertex of B. Thus $|V(T')| < 2s(G)$. ■

The rest of this paper is organized as follows: In Section 2, we discuss the complexity of deciding whether a given connected graph has safe number at most a given integer t, In Section 3, we focus on the safe number of trees. In Section 4, we investigate lower bounds on the connected safe number of a graph with given minimum degree or maximum degree.

2. Complexity

We consider the following decision problems:

SAFE SET
INSTANCE: A graph G and an integer t.
QUESTION: Does there exist a safe set $S \subseteq V(G)$ with $1 \leq |S| \leq t$?

CONNECTED SAFE SET
INSTANCE: A graph G and an integer t.
QUESTION: Does there exist a connected safe set $S \subseteq V(G)$ with $1 \leq |S| \leq t$?

The complexity of these problems is determined as a consequence of the following construction and proposition. Let G be a connected graph with $V(G) = \{v_1, v_2, \dots, v_n\}$. Let H_G be graph constructed from G, a new vertex v_{n+1}, and $n + 1$ vertex-disjoint paths $P_i = p_{i_1} p_{i_2} \dots p_{i_{n-k}}$, $1 \leq i \leq n + 1$, each of order $n - k$, by joining v_{n+1} to each of $v_1, v_2, \dots, v_n$, and then joining v_i and p_{i_1} for $i = 1, 2, \dots, n + 1$. Note that H_G is a connected graph of order $(n + 1)(n - k + 1)$, and

Proposition 3. *Let G be a connected graph with $V(G) = \{v_1, v_2, \dots, v_n\}$. The following two statements are equivalent:*

(i) $\alpha(G) \geq k$
(ii) *H_G has a connected safe set of size $n - k + 1$.*

Proof. (i) ⇒ (ii): Let X be an independent set of size k in G, and let $S = (V(G) - X) \cup \{v_{n+1}\}$. Thus, $|S| = n - k + 1$ and $H_G[S]$ is connected. Since every component of $H_G - S$ has size at most $n - k + 1$, the set S forms a connected safe set of size $n - k + 1$. Thus (i) implies (ii).

(ii) ⇒ (i): Let S' be a connected safe set of H_G with size $n - k + 1$. Note that $v_{n+1} \in S'$ since otherwise $H_G - S'$ contains a component of size greater than $n - k + 1$. The set $S' \cap V(G)$ contains at most $n - k$ vertices of G. Note that every vertex of $G - S'$ belongs to a component of $H_G - S'$ of size at least $n - k + 1$. Since S' is a safe set of H_G, each component of $H_G - S'$ contains at most one vertex of G. Consequently, the vertices of $G - S'$ are an independent set of size at least k. Thus (ii) implies (i). ■

Corollary 1. *CONNECTED SAFE SET is NP-complete.*

Since a smallest safe set in the graph H_G of Proposition 3 must contain the vertex v_{n+1}, and hence induce a connected subgraph, we also have the following:

Corollary 2. *SAFE SET is NP-complete.*