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Resistance distances and Kirchhoff index of graphs with an involution

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ABSTRACT

Motivated by the work of Zhang and Yan (2009), this paper considers the problem of computing resistance distances and Kirchhoff index of graphs with an involution. We show that if G is a weighted graph with an involution, then the resistance distance and the Kirchhoff index of G can be expressed in terms of parameters of two weighted graphs with a smaller size. As applications, we compute resistance distances and Kirchhoff indices of double graphs, the almost-complete graph and the almost-complete bipartite graph.

1. Introduction

Suppose that $G = (V(G), E(G))$ is a connected edge-weighted graph (for convenience, we say a connected weighted graph) and the weight of each edge e of G is denoted by $w(e)$ which is a real number. The weight $w(T)$ of a spanning tree T in G is defined as the product of weights of edges in T, i.e.,

$$w(T) = \prod_{e \in E(T)} w(e).$$

The sum of weights of spanning trees of G is denoted by $t(G)$. Hence

$$t(G) = \sum_{T} w(T),$$

where the summation is over all spanning trees of G.

For any connected weighted graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. If $w(e) > 0$ for every edge $e \in E(G)$, then we can view it as an electrical network with $w(e)$ as the conductance of edge e (i.e., $1/w(e)$ is the resistance of e). We denote the effective resistance of the network between $u, v \in V(G)$ by $R_G(u, v)$. Then we have the well-known combinatorial formula involving spanning trees [17]:

$$R_G(u, v) = \frac{t(G^{uv})}{t(G)}, \tag{1.1}$$

where G^{uv} denotes the graph obtained from G by identifying the two vertices u and v. The effective resistance is a distance function (or metric) on graphs, which was first proved by Sharpe [18], and then rediscovered by Gvishiani and Gurvich [10],

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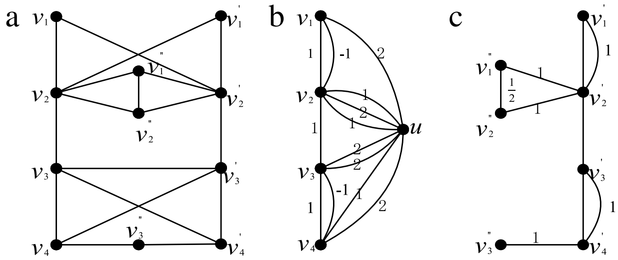

Fig. 1. (a) A connected graph G with an involution f such that $f(v_i) = v_i'$, $f(v_i') = v_i$, and $f(v_j'') = v_j''$ for $1 \le i \le 4$, $1 \le j \le 3$; where the left part G_L (respectively right part G_R) of G is the subgraph induced by $\{v_1, v_2, v_3, v_4\}$ (respectively $\{v_1', v_2', v_3', v_4'\}$) and the center part G_C of G is the subgraph induced by $\{v_1'', v_2'', v_3''\}$. (b) The weighted graph $\overline{G_L}$. (c) The weighted graph $\overline{G_R}$.

also Klein and Randić [14]. In [14], Klein and Randić named this distance function the "resistance distance" and introduced the sum of resistance distances between all pairs of vertices of G,

$$Kf(G) = \sum_{u,v \in V(G)} R_G(u, v), \tag{1.2}$$

which was named "the Kirchhoff index" of G.

The resistance distance and Kirchhoff index attracted extensive attention due to their wide applications in physics, chemistry and others. Besides a number of well-known techniques developed by electrical engineers, such as, series and parallel principles, star-triangle transformation (or general star-mesh transformation) [1,16,19], various formulas for computing resistance distances have been derived, including algebraic formulas, probabilistic formulas, combinatorial formulas, sum rules, recursion formula and so on, for more details, see [12,20,23] and references therein. Using the above techniques and formulas, resistance distances and Kirchhoff index for many interesting (classes of) graphs have been computed, for example, some fullerene graphs [7], circulant graphs [22], distance-regular graphs [13] and so on. In this paper, we shall consider resistance distances and Kirchhoff index of weighted graphs with an involution. This work is inspired mostly by Fuji Zhang and Weigen Yan's result in [21], where they expressed the sum of weights of spanning trees of G in terms of the product of that of two weighted graphs with a smaller size. So the following definitions and notation follow basically from [21].

Suppose that G is a connected weighted graph. If the underlying graph of G has an involution f and for every $e \in E(G)$ we have $w(e) = w(f(e))$, that is, the weighted function w on the edges is constant on the orbits of f, then we say that G is a weighted graph with an involution f (or f preserves weights), see Fig. 1(a) for an illustration example. From the definition, for a connected weighted graph $G = (V(G), E(G))$ with an involution f, $V(G)$ can be partitioned into $V(G) = V_L \cup V_C \cup V_R$, where $V_L = \{v_1, v_2, \dots, v_s\}$, $V_C = \{v_1'', v_2'', \dots, v_n''\}$, $V_R = \{v_1', v_2', \dots, v_s'\}$ (V_C can be the empty set) and $f(v_i) = v_i'$, $f(v_i') = v_i$, and $f(v_j'') = v_j''$ for $1 \le i \le s$, $1 \le j \le n$. Let G_L and G_R denote the two isomorphic weighted subgraphs of G induced by V_L and V_R, respectively. Similarly, let G_C denote the weighted subgraph of G induced by V_C. We can partition the edge set of G as follows:

$$E(G) = E(G_L) \cup E(G_R) \cup E(G_C) \cup E(G_L - G_R) \cup E(G_L - G_C) \cup E(G_R - G_C),$$

where $E(G_L - G_R)$ denotes the set of edges in G between G_L and G_R, and $E(G_L - G_C)$ (or $E(G_R - G_C)$) is the set of edges in G between G_L and G_C (or between G_R and G_C).

Now two new weighted graphs $\overline{G_L}$ and $\overline{G_R}$ from G are constructed as follows. Let $\overline{G_L}$ be the weighted graph obtained from G_L by the following four procedures (see Fig. 1(b)):

1. Add a new vertex u to G_L.
2. For each edge $e = (v_i, v_j') \in E(G_L - G_R)$ with weight $w(e)$, add an edge (u, v_i) in $\overline{G_L}$ with weight $2w(e)$.
3. For each pair edge $e = (v_i, v_j')$ and $e' = (v_j, v_i')$ (where $i \ne j$) in $E(G_L - G_R)$, add an edge (v_i, v_j) in $\overline{G_L}$ with weight $-w(e)$.
4. For each edge $e = (v_i, v_j'') \in E(G_L - G_C)$, add an edge (u, v_i) in $\overline{G_L}$ with weight $w(e)$.

Let $\overline{G_R}$ be the weighted graph obtained from the weighted subgraph of G induced by $V_C \cup V_R$ by the following two procedures (see Fig. 1(c)):

1′ Reduced the weight of each edge in G_C by half.
2′ For pair of edges $e = (v_i, v_j')$ and $e' = (v_j, v_i')$ (where $i \ne j$) in $E(G_L - G_R)$, add an edge (v_i', v_j') with weight $w(e)$.

Then we have the following result.

Theorem 1.1 ([21]). *Suppose that $G = (V(G), E(G))$ is a weighted graph with an involution f and $\overline{G_L}$, $\overline{G_R}$ are defined as above. Then the sum of weights of spanning trees of G is given by*

$$t(G) = 2^{n-1} t(\overline{G_L}) t(\overline{G_R}),$$

where n is the number of vertices of G_C.

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