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Note

Remoteness and distance eigenvalues of a graph

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ABSTRACT

Let G be a connected graph of order n with diameter d. Remoteness ρ of G is the maximum average distance from a vertex to all others and $\partial_1 \geq \cdots \geq \partial_n$ are the distance eigenvalues of G. Aouchiche and Hansen (0000), Aouchiche and Hansen conjectured that $\rho + \partial_3 > 0$ when $d \geq 3$ and $\rho + \partial_{\lfloor \frac{7d}{8} \rfloor} > 0$. In this paper, we confirm these two conjectures. Furthermore, we give lower bounds on $\partial_n + \rho$ and $\partial_1 - \rho$ when $G \ncong K_n$ and the extremal graphs are characterized.

1. Introduction

Throughout this paper we consider simple, undirected and connected graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$, $|E(G)| = m$. Also let d_i be the degree of the vertex $v_i \in V(G)$. For $v_i, v_j \in V(G)$, the distance between vertices v_i and v_j is the length of a shortest path connecting them in G, denoted by $d_G(v_i, v_j)$ or d_{ij}. The diameter of a graph is the maximum distance between any two vertices of G. Let d be the diameter of G. The transmission $Tr(v_i)$ (or D_i or $D_i(G)$) of vertex v_i is defined to be the sum of distances from v_i to all other vertices, that is,

$$Tr(v_i) = \sum_{v_j \in V(G)} d(v_i, v_j).$$

We call G is transmission regular if $Tr(v_1) = \cdots = Tr(v_n)$ (or $D_1 = \cdots = D_n$). The remoteness of G is denoted by

$$\rho = \max_{v \in V} \frac{Tr(v)}{n-1}.$$

In [2], Aouchiche and Hansen compared the graph invariant ρ to the radius, average eccentricity, average distance and the distance eigenvalues. Most of the results were proved, but a few of them remained as conjectures. Recently, some authors including one of the present author solved several conjectures related to remoteness (see [3,4]).

The distance matrix of G, denoted by $D(G)$ or simply by D, is the symmetric real matrix with (i, j)-entry being $d_G(v_i, v_j)$ (or d_{ij}). The distance eigenvalues (resp. distance spectrum) of G, denoted by

$$\partial_1(G) \geq \partial_2(G) \geq \cdots \geq \partial_n(G).$$

When $G \ncong K_n$, Lin et al. [8] showed that $\partial_n(G) \leq -2$ with equality holding if and only if G is a complete multipartite graph. Later, Lin [7] generalized the result and proved that $\partial_n(G) \leq -d$ with equality holding if and only if G is a complete

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multipartite graph. Recently, the distance matrix of a graph has received increasing attention, see [6,8,10,11]. For more results on the distance matrix, you can refer to the excellent survey [1]. Aouchiche and Hansen [2] posed the following two conjectures which are related to the remoteness and distance eigenvalues.

Conjecture 1. *Let G be a graph on $n \geq 4$ vertices with diameter $d \geq 3$, remoteness ρ and distance eigenvalues $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. Then*

$$\rho + \partial_3 > 0.$$

In this paper, we confirm the conjecture and give a lower bound on $\rho + \partial_3$ when d is equal to 2.

Theorem 1.1. *Let G be a connected graph of order $n \geq 4$ with diameter d, remoteness ρ and distance eigenvalues $\partial_1 \geq \cdots \geq \partial_n$. Then we have the following statements.*

(i) *If $d = 2$, then*

$$\rho + \partial_3 \geq \frac{\left\lceil \frac{n}{2} \right\rceil - 2}{n-1} - 1,$$

with equality holding if and only if $G \cong K_{n_1,n_2}$.

(ii) *If $d \geq 3$, then*

$$\rho + \partial_3 > \frac{d}{2} - 1.2.$$

Conjecture 2. *Let G be a connected graph of order $n \geq 4$ with diameter d, remoteness ρ and distance spectrum $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. Then*

$$\rho + \partial_{\left\lfloor \frac{7d}{8} \right\rfloor} > 0.$$

In the paper, we confirm the conjecture too. We state it as the following theorem.

Theorem 1.2. *Let G be a connected graph of order $n \geq 4$ with diameter d, remoteness ρ and distance spectrum $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. Then*

$$\rho + \partial_{\left\lfloor \frac{7d}{8} \right\rfloor} > 0.$$

In Section 3, we give lower bounds on $\partial_n + \rho$ and $\partial_1 - \rho$ when $G \ncong K_n$ and the extremal graphs are characterized.

2. Proofs

Lemma 2.1 ([7])**.** *Let G be a connected graph of order n with diameter d. Then*

$$\partial_n \geq -d,$$

with equality holding if and only if G is complete multipartite graph.

Lemma 2.2. *Let G be a connected graph of order n with diameter d and remoteness ρ. Then*

$$\rho > \frac{d}{2}.$$

Proof. Suppose that $P_{d+1} = v_1 v_2 \cdots v_{d+1}$ is a diametral path of G. Then for each vertex $v \in V(G) \setminus V(P_{d+1})$, we have

$$d(v, v_1) + d(v, v_{d+1}) \geq d.$$

Therefore,

$$\begin{aligned} \rho &\geq \frac{\max\left\{tr(v_1),\ tr(v_{d+1})\right\}}{n-1} \\ &\geq \frac{tr(v_1) + tr(v_{d+1})}{2\,(n-1)} \\ &\geq \frac{2(1 + \cdots + d) + d(n-d-1)}{2\,(n-1)} \\ &= \frac{n\,d}{2\,(n-1)} > \frac{d}{2}. \end{aligned}$$

This completes the proof. □