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Note

Bicyclic graphs with maximal edge revised Szeged index☆

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ABSTRACT

The edge revised Szeged index $Sz_e^*(G)$ is defined as $Sz_e^*(G) = \sum_{e=uv\in E}(m_u(e) + m_0(e)/2)(m_v(e) + m_0(e)/2)$, where $m_u(e)$ and $m_v(e)$ are, respectively, the number of edges of G lying closer to vertex u than to vertex v and the number of edges of G lying closer to vertex v than to vertex u, and $m_0(e)$ is the number of edges equidistant to u and v. In this paper, we give an upper bound of the edge revised Szeged index for a connected bicyclic graphs with size $m \geq 5$, that is,

$$Sz_e^*(G) \leq \begin{cases} (m^3 - 4m + 16)/4, & \text{if } m \text{ is odd,} \\ (m^3 - 4m + 18)/4, & \text{if } m \text{ is even} \end{cases}$$

with equality if and only if G is the graph obtained from the cycle C_{m-2} by duplicating a single vertex.

1. Introduction

All graphs considered in this paper are finite, undirected and simple. We refer the readers to [2] for terminology and notations. Let C_n denote the cycle on n vertices. Let G be a connected graph with vertex set V and edge set E. For $u, v \in V$, $d(u, v)$ denotes the distance between u and v. The *Wiener index* of G is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V} d(u, v).$$

This topological index has been extensively studied in the mathematical literature; see, e.g., [8,10]. Let $e = uv$ be an edge of G, and define three sets as follows:

$$N_u(e) = \{w \in V : d(u, w) < d(v, w)\},$$
$$N_v(e) = \{w \in V : d(v, w) < d(u, w)\},$$
$$N_0(e) = \{w \in V : d(u, w) = d(v, w)\}.$$

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Thus, $\{N_u(e), N_v(e), N_0(e)\}$ is a partition of the vertices of G with respect to e. The number of vertices of $N_u(e), N_v(e)$ and $N_0(e)$ are denoted by $n_u(e), n_v(e)$ and $n_0(e)$, respectively. A long time known property of the Wiener index is the formula [9,18]:

$$W(G) = \sum_{e=uv\in E} n_u(e)n_v(e),$$

which is applicable for trees. Using the above formula, Gutman [6] introduced a graph invariant named the *Szeged index* as an extension of the Wiener index and defined it by

$$Sz(G) = \sum_{e=uv\in E} n_u(e)n_v(e).$$

Randić [15] observed that the Szeged index does not take into account the contributions of the vertices at equal distances from the endpoints of an edge, and so he conceived a modified version of the Szeged index which is named the *revised Szeged index*. The revised Szeged index of a connected graph G is defined as

$$Sz^*(G) = \sum_{e=uv\in E} \left(n_u(e) + \frac{n_0(e)}{2}\right)\left(n_v(e) + \frac{n_0(e)}{2}\right).$$

Some properties and applications of these topological indices have been reported in [1,4,12–14,16,19].

Given an edge $e = uv \in E$, the distance between the edge e and the vertex x, denoted by $d(e, x)$, is defined as

$$d(e, x) = \min\{d(u, x), d(v, x)\}.$$

Similarly, the sets $M_0(e), M_u(e)$ and $M_v(e)$ are defined to be the set of edges equidistant from u and v, the set of edges whose distance to vertex u is smaller than the distance to vertex v and the set of edges closer to v than u, respectively. The number of edges of $M_u(e), M_v(e)$ and $M_0(e)$ are denoted by $m_u(e), m_v(e)$ and $m_0(e)$, respectively. Then, The edge Szeged index [7], and edge revised Szeged index [5] of G are defined as follows:

$$Sz_e(G) = \sum_{e=uv\in E} m_u(e)m_v(e),$$

$$Sz_e^*(G) = \sum_{e=uv\in E} \left(m_u(e) + \frac{m_0(e)}{2}\right)\left(m_v(e) + \frac{m_0(e)}{2}\right).$$

Results on edge Szeged index can be found in [3,11,17]. In [5], they determined the n-vertex unicyclic graphs with the largest and the smallest revised edge Szeged indices. In this paper, we give an upper bound of the edge revised Szeged index for a connected bicyclic graphs, and also characterize those graphs that achieve the upper bound.

Theorem 1.1. *Let G be a connected bicyclic graph of size $m \geq 5$. Then*

$$Sz_e^*(G) \leq \begin{cases} (m^3 - 4m + 16)/4, & \text{if } m \text{ is odd}, \\ (m^3 - 4m + 18)/4, & \text{if } m \text{ is even} \end{cases}$$

with equality if and only if G is the graph obtained from the cycle C_{m-2} by duplicating a single vertex.

2. Main results

For convenience, let B_m be the graph obtained from the cycle C_{m-2} by duplicating a single vertex (see Fig. 1), where m is the size of B_m. It is easy to check that

$$Sz_e^*(B_m) = \begin{cases} (m^3 - 4m + 16)/4, & \text{if } m \text{ is odd}, \\ (m^3 - 4m + 18)/4, & \text{if } m \text{ is even}. \end{cases}$$

So, we are left to show that for any connected bicyclic graph G_m of size m, other than B_m, $Sz^*(G_m) < Sz^*(B_m)$. Using the fact that $m_u(e) + m_v(e) + m_0(e) = m$, we have

$$\begin{aligned} Sz_e^*(G_m) &= \sum_{e=uv\in E} \left(m_u(e) + \frac{m_0(e)}{2}\right)\left(m_v(e) + \frac{m_0(e)}{2}\right) \\ &= \sum_{e=uv\in E} \left(\frac{m + m_u(e) - m_v(e)}{2}\right)\left(\frac{m - m_u(e) + m_v(e)}{2}\right) \\ &= \sum_{e=uv\in E} \frac{m^2 - (m_u(e) - m_v(e))^2}{4}. \end{aligned}$$