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Implementing generating functions to obtain power indices with coalition configuration

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ABSTRACT

We consider the Banzhaf–Coleman and Owen power indices for weighted majority games modified by a coalition configuration. We present calculation algorithms of them that make use of the method of generating functions. We programmed the procedure in the open language R and it is illustrated by a real life example taken from social sciences. To finish, we generalize the algorithms to a wider class of games.

1. Introduction

In a cooperative game with transferable utility, also called TU-game, agents have mechanisms that allow them to make binding agreements. The utility derived from the cooperation of any set of agents (or players) can be transferred and divided in any way between them. One of the main objectives of cooperative game theory is the study of values, that is, distribution rules that assign to each game a vector, so that each coordinate vector represents the payment allocated to each player. There are numerous values, two of which, widely used, are the Shapley value (Shapley, [20]) and the Banzhaf value (Banzhaf, [9]).

An important family of TU-games is formed by simple games, which have interesting applications, especially in the field of political science. Within the family of simple games, weighted majority games play an important role. For example, a Parliament can be seen as a weighted majority game, assuming that players are the political parties and decisions are taken by majority.

When working with simple games, instead of talking about value, it is often used the term power index as simple games are usually used as models of decision-making bodies in which agreements are taken by voting. The interest in these games usually focuses on knowing the power or influence having a player in the final result. In this context, the Shapley value and the Banzhaf value are renamed Shapley–Shubik power index (Shapley and Shubik, [21]) and Banzhaf–Coleman power index (Banzhaf, [9] and Coleman, [12]), respectively. Numerous works in the literature are devoted to the definition, study, calculation, and applications of power indices.

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The power indices can be obtained in an exact or approximated manner by using different tools. One of the most commonly used is given by the so called generating functions. This, in the case of weighted majority games, is based on the use of a technique of combinatorial analysis. Roughly speaking, a generating function is a polynomial that allows to enumerate the set of possible coalitions, while having control about their respective weights. This is very useful because it allows to obtain the exact value of the indices, even in games with many players, making use of algorithms that can be programmed with a computer language. This technique was used by David G. Cantor (1962) (cf. Lucas, [15]) for the Shapley–Shubik power index and by Brams and Affuso [11] for calculating the Banzhaf–Coleman power index.

In a more general model of TU-games with coalition structure, a partition is considered over the set of players, because they could have some preferences to join others motivated by some reasons as ideological features, or geographic location. For this reason, the Owen value (Owen, [18]) was proposed for TU-games with coalition structure, which generalizes the Shapley value, and the Banzhaf–Owen power index (Owen, [19]) was proposed for simple games with coalition structure as an extension of the Banzhaf–Coleman power index. Alonso-Meijide and Bowles [7] used generating functions to calculate the Owen and Banzhaf–Owen values for weighted majority games.

Nonetheless, this model might not be appropriate in some situations in which players could prefer to join some players for some reasons and join some others for other reasons. Let us illustrate this situation with an example taken from Andjiga and Courtin [8]: Consider the diplomatic relations among countries. In real life, countries are organized into international coalitions not necessarily disjoint. For example, France and Spain, among others, belong to the European Union and the North Atlantic Treaty Organization (NATO), while The United States belongs to NATO and NAFTA (a coalition with Mexico and Canada). For this reason, the more general model of games with coalition configuration was introduced. In this model, it is considered a cover of the set of players instead of a partition of it. Albizuri and Aurrekoetxea [1] and Albizuri, Aurrekoetxea, and Zarzuelo [2] propose the generalized Banzhaf–Coleman index and the configuration value,[1] for the models of simple games and TU-games, respectively, with a coalition configuration. These rules generalize the Banzhaf–Coleman power index and the Owen value, respectively.

In the current paper, we apply the method of generating functions to calculate the generalized Banzhaf–Coleman index and the configuration value for the family of weighted majority games with coalition configuration in such a way that we extend the results of Lucas [15], Brams and Affuso [11], and Alonso-Meijide and Bowles [7]. We illustrate the algorithms with a small numerical example and another one taken from real life. We also show an R code that implements the new algorithms. Moreover, we generalize the algorithms to the class of weighted multiple majority games that model many voting schemes.

We start the work with some preliminaries.

2. Preliminaries

A *cooperative game with transferable utility* (shortly a *TU-game*) is a pair (P, f), where $P = \{1, \ldots, p\}$ is a *set of players* and f is a real function (the *characteristic function*) which allocates to each coalition $T \subseteq P$ a real value. This value can be interpreted as the payoff obtained by each subset of players as the result of the cooperation among its members.

We say that a TU-game is *simple* if f only takes the values 0 or 1, $f(P) = 1$, and the game is monotonic, namely, $f(T) \leq f(S)$ if $T \subseteq S \subseteq P$. We represent for $W(f)$ the set of all *winning coalitions* of the game (P, f), it means, $W(f) = \{T \subseteq P : f(T) = 1\}$. In this paper, we will work with a particular class of simple games, the *weighted majority games*, in which exists a set of *weights* $w_1, \ldots, w_p$, with $w_i \in \mathbb{N} \cup \{0\}$ for $i = 1, \ldots, p$, and a *quota* $q \in \mathbb{N}$ $(q > 0)$, such that for each $T \subseteq P$, $f(T) = 1$ if and only if $w(T) \geq q$ where $w(T) = \sum_{i \in T} w_i$. We will represent a weighted majority game by $[q; w_1, \ldots, w_p]$.

Given a sequence of real numbers, $\{a_n\}_{n \in \mathbb{N}}$, we can associate the following series:

$$f_a(x) = \sum_{j \geq 0} a_j x^j.$$

This series is called the *generating function* of the sequence a_n, and may be finite or infinite.

In this paper, we will work with generating functions of several variables, which can be written as $f_a(x, y, z) = \sum_{j \geq 0} \sum_{k \geq 0} \sum_{l \geq 0} a_{j,k,l} x^j y^k z^l$ where $a_{j,k,l}$ represents a number depending on j, k, and l.

Let us consider a set of players $P = \{1, \ldots, p\}$. A *coalition configuration* $C = \{C_1, \ldots, C_c\}$ is a cover of P, it is, a collection of non-empty subsets of P such that $\bigcup_{k=1}^{c} C_k = P$. As a consequence of the fact that each player could be simultaneously in more than one element of C, for each $i \in P$, we will consider $C^i = \{C_k \in C : i \in C_k\}$ as the set of elements of C in which player i is in. A *TU-game with coalition configuration* is a triple $(P, f; C)$ where (P, f) is a TU-game and C a coalition configuration over P. Let us denote by $SC(P)$ the family of simple games with coalition configuration and players set P.

Albizuri and Aurrekoetxea [1] generalize the Banzhaf–Coleman index to simple games with coalition configuration, which we call here *generalized Banzhaf–Coleman index*, and it is defined for each simple game with coalition configuration $(P, f; C)$

[1] We use for these values the names proposed in Albizuri and Aurrekoetxea [1] and Albizuri, Aurrekoetxea, and Zarzuelo [2].