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One pile misère bounded Nim with two alliances

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ABSTRACT

The game of *n-person one-pile bounded Nim with two alliances* is investigated: Given an integer $m \geq 1$ and a pile of counters, each player is allowed to remove ℓ counters from the pile, where $\ell \in \{1, 2, \ldots, m\}$. Suppose that $n \geq 2$ players form two alliances and that each player is in exactly one alliance. Also assume that each player will support his alliance's interests.

Under misère play convention, all unsafe positions of two alliances are determined for some structures of two alliances. We also point out that some conclusions given by A.R. Kelly are not correct. Moreover, we present a possible explanation for Kelly's inaccurate conclusions.

1. Introduction

Combinatorial game theory is a branch of mathematics devoted to studying the optimal strategy in perfect-information games where typically two players are involved. In a 2-person perfect information game two players alternately move until one of them is unable to move at his turn. Among the games of this type are *Nim* [2,4,8,10], *End-Nim* [1,9], *Wythoff's game* [7,6,15], *a-Wythoff's game*, (s, t)-*Wythoff's game* [14,16], etc. There are two conventions: in *normal play convention*, the player first unable to move is the loser (his opponent the winner); in *misère play convention*, the player first unable to move is the winner (his opponent the loser). The positions from which the previous player can win regardless of the opponent's moves are called *P-positions* and those from which the next player can win regardless of the opponent's moves are called *N-positions*. The theory of such games can be found in [3,5].

1.1. 2-person Nim

The game of Nim is well known. The game is played with piles of counters. The two players take turns removing any positive integer of counters from any one pile. Under normal play convention, Bouton's analysis of Nim [4] showed that the *P*-positions are those for which nim-addition on the sizes of the piles is 0, and the *N*-positions are those for which nim-addition on the sizes of the piles is greater than 0. In the same paper, all *P*-positions of Nim were determined under misère play convention.

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1.2. n-person Nim

During the last few years, the theory of 2-person perfect information games has been promoted to an advanced level. Naturally it is of interest to generalize as much as possible of the theory to n-person games. In 2-person perfect information games, one can always talk about what the outcome of the game *should* be, when each player plays it right, i.e., when each player adopts an optimal strategy. But when there are more than two players, it may not make sense to talk about the same thing. For instance, it may so happen that one of the players can help any of the players to win, but anyhow, he himself has to lose. So the outcome of the game depends on how the group coalitions are formed among the players. In previous literatures, two directions were investigated: *n-person without alliance* and *n-person with two alliances*.

1.2.1. N-person Nim without alliance

The game *n-person Nim without alliance* was introduced in [13]: The n players are $P_1, P_2, \ldots, P_n$, according to the initial order of turns. The players rotate turns moving counters from *any* one pile of $(c_1, c_2, \ldots, c_p)$. The game is ended when any player is unable to move at his turn. Naturally under normal play convention, we define the loser to be the player unable to move. If that player is P_m, say, we assign a different rank to each player, ranging from bottom to top in the order of $P_m, P_{m+1}, \ldots, P_n, P_1, P_2, \ldots, P_{m-1}$. In particular, the last player able to move is the *top* winner. Under these rules, the rank of any one player automatically determines the ranks for all. For this reason, it makes sense to say what the outcome of the game should be when each player adopts an optimal strategy toward his own highest possible rank.

1.2.2. N-person Nim with two alliances

The game of *n-person one-pile bounded Nim with two alliances* was investigated in [12,11]: Given an integer $m \geq 1$ and a pile of counters, suppose that $n \geq 2$ players form two alliances and that each player is in exactly one alliance. Also assume that each player will support his alliance's interests. Each player is allowed to remove ℓ counters from the pile, where $\ell \in \{1, 2, \ldots, m\}$. Under misère play convention, the alliance which takes the last counter is the loser (the other alliance is the winner); under normal play convention, the alliance which takes the last counter is the winner (the other alliance is the loser).

A position is defined to be *an unsafe position of one alliance* if the game begins from this position and no matter what move this alliance makes, when the other alliance plays optimally, this alliance must lose. In [12,11], under misère play convention, Annela R. Kelly gave all unsafe positions of two alliances for some special structures of two alliances. However, we find that some conclusions given by A. R. Kelly are not correct.

1.3. Our games and results

Definition 1 (*General Structure of Two Alliances*)**.** Given $n \geq 2$ players $P_1, P_2, \ldots, P_n$ in an initial order of turns. Suppose that these n players form two alliances and that each player is in exactly one alliance. Generally, n players are divided into p consecutive parts:

(1) If $p = 2k \geq 2$ then we represent p consecutive parts by $A_1^1, A_2^1, A_1^2, A_2^2, \ldots, A_1^k, A_2^k$. The players in $A_1^1, A_1^2, \ldots, A_1^k$ form alliance A_1, and those in $A_2^1, A_2^2, \ldots, A_2^k$ form alliance A_2. For brevity, we call it "**Alliance**-[k; k]".
(2) If $p = 2k + 1 \geq 3$ then we represent p consecutive parts by $A_1^1, A_2^1, A_1^2, A_2^2, \ldots, A_1^k, A_2^k, A_1^{k+1}$. The players in $A_1^1, A_1^2, \ldots, A_1^k, A_1^{k+1}$ form alliance A_1, and those in $A_2^1, A_2^2, \ldots, A_2^k$ form alliance A_2. For brevity, we call it "**Alliance**-[$k + 1$; k]".
(3) By s_i we denote the number of players in A_1^i, and t_i the number of players in A_2^i.

For example, we consider $n = 7$ players $P_1, P_2, \ldots, P_6, P_7$:

(1) Assume that P_1, P_2, P_5, P_6 form A_1, and P_3, P_4, P_7 form A_2. By Definition 1, alliance A_1 consists of two consecutive parts $A_1^1 = \{P_1, P_2\}$ and $A_1^2 = \{P_5, P_6\}$; alliance A_2 consists of two consecutive parts $A_2^1 = \{P_3, P_4\}$ and $A_2^2 = \{P_7\}$. The structure of two alliances is "*Alliance*-[2; 2]" and $s_1 = s_2 = 2$ and $t_1 = 2, t_2 = 1$.
(2) Assume that P_1, P_2, P_5, P_6, P_7 form A_1, and P_3, P_4 form A_2. By Definition 1, alliance A_1 consists of two consecutive parts $A_1^1 = \{P_1, P_2\}$ and $A_1^2 = \{P_5, P_6, P_7\}$; alliance A_2 consists of one consecutive part $A_2^1 = \{P_3, P_4\}$. The structure of two alliances is "*Alliance*-[2; 1]" and $s_1 = 2, s_2 = 3$ and $t_1 = 2$.
(3) Assume that P_1, P_3, P_5, P_7 form A_1, and P_2, P_4, P_6 form A_2. By Definition 1, alliance A_1 consists of four consecutive parts $A_1^1 = \{P_1\}, A_1^2 = \{P_3\}, A_1^3 = \{P_5\}$ and $A_1^4 = \{P_7\}$; alliance A_2 consists of three consecutive parts $A_2^1 = \{P_2\}, A_2^2 = \{P_4\}$ and $A_2^3 = \{P_6\}$. The structure of two alliances is "*Alliance*-[4; 3]" and $s_1 = s_2 = s_3 = s_4 = 1$ and $t_1 = t_2 = t_3 = 1$.

Definition 2. (1) **One-pile misère bounded Nim with Alliance**-[k; k] (denoted by $\Gamma_{k,k}^m$): Given two integers $m \geq 1$ and $n \geq 2$. In one-pile misère bounded Nim with n players, all players form **Alliance**-[k; k] and each player will support his alliance's interests. Each player is allowed to remove ℓ counters from the pile, where $\ell \in \{1, 2, \ldots, m\}$. The alliance which takes the last counter is the loser (the other alliance is the winner).
(2) Similarly, we define **One-pile misère bounded Nim with Alliance**-[$k + 1$; k], denoted by $\Gamma_{k+1,k}^m$.