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Graph odometry

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ABSTRACT

We address the problem of determining edge weights on a graph using non-backtracking closed walks from a vertex. We show that the weights of all of the edges can be determined from any starting vertex exactly when the graph has minimum degree at least three. We also determine the minimum number of walks required to reveal all edge weights.

1. Introduction

The present paper addresses the problem of recovering the edge-weights of a graph, given the weight of every non-backtracking walk from a particular vertex. The problem of recovering the edge-weights from some limited amount of information about a graph fits naturally into the area of graph reconstruction, which is well-known and extensively studied [7,2]. On the other hand, the restriction of the information to the weights of all non-backtracking walks from a vertex seems somewhat unnatural, primarily the *non-backtracking* assumption. However, in the field of low-dimensional topology and geometry, graphs that prohibit backtracking are not uncommon. A *weighted train track* is a weighted graph embedded on a differentiable surface, with the property that at every vertex, the edges are partitioned into two sets. Every pair of edges incident to a vertex with one from each set of the partition must have a common tangent at the vertex [4,6]. Train tracks have more properties and restrictions than this, but for the present purposes, the above definition suffices. If one were to actually travel on such a train track graph, then at each vertex, the traveler could choose only edges from the opposing set to continue on, ruling out the edge most recently traversed. Thus any walk taken on such a graph should be *non-backtracking*, in that an edge cannot be immediately traversed in the opposite direction. The condition of non-backtracking on a walk is *less* strict than the allowed walks in the train track scenario, in that for a train track graph, multiple edges are disallowed including the most recent one traversed. Nonetheless, the notion of allowing only non-backtracking walks is interesting as a relaxation of this property.

The authors claim no expertise in the area of low-dimensional topology and geometry, and instead focus on a purely graph-theoretic formulation of the following problem. Given access to the weight (i.e., the sum of the weights of the traversed edges) of every closed, non-backtracking walk from a fixed vertex of a graph, can each of individual edge weights be determined?

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2. Preliminaries

Throughout the paper, G is assumed to be a finite, undirected graph. Let $V(G)$ denote the vertex set of G and let $E(G)$ denote the edge set of G. Unless otherwise noted, we follow the notation of Diestel [1]. Here we recall several definitions which will be used throughout the document. Given a connected graph G, a *cut vertex* v of G is a vertex such that $G \setminus \{v\}$ is not connected. A *block* B of G is a maximal subgraph of G which contain no cut vertex (in B). Hence any block of G is either a maximal 2-connected component, a bridge, or an isolated vertex. Any two blocks intersect in at most one vertex which is a cut vertex in G. The *block graph* of G is the graph whose vertices are the blocks of G with an edge between two blocks if and only if they share a vertex. The block graph of G is a tree, as if there were a cycle in the block graph the union of these blocks would result in a larger block, contradicting maximality. Further, every edge of G lies in some block, and so G is the union of its blocks.

Define a *walk* in G to be a sequence $W = \{v_j, v_{j+1}, \dots, v_k\} \subseteq V(G)$ with $\{v_i, v_{i+1}\} \in E(G)$ for $j \le i \le k-1$. We call a walk a *non-backtracking walk* if we require $v_i \ne v_{i+2}$ for $j \le i \le k-2$. Finally, we call a (non-backtracking) walk *closed* if $v_j = v_k$. Notice that when viewed as a sequence (i.e., order matters), this walk is "anchored" at v_j in the following sense: The walk $\{a, b, c, d, b, a\}$ is non-backtracking, whereas the walk $\{b, c, d, b, a, b\}$ is backtracking. While these walks are isomorphic when viewed as subgraphs, they are considered different in our application. For walks $W_1 = \{v_j, v_{j+1}, \dots, v_k\}$, $W_2 = \{v_k, v_{k+1}, \dots, v_\ell\}$ with $j < k < \ell$, define the binary operation $\circ$ as follows: $W_1 \circ W_2 := \{v_j, v_{j+1}, \dots, v_k, v_{k+1}, \dots, v_\ell\}$, i.e., concatenation of the walks. Define the unary operation $\bar{\cdot}$ as follows: $\overline{W_1} := \{\hat{v}_j = v_k, \hat{v}_{j+1} = v_{k-1}, \dots, \hat{v}_k = v_j\}$, i.e., reversal of the indices.

Now let $F : E(G) \to \Omega$ be a weight function of the edge set. We normally take Ω to be some real interval, although any field or $\mathbb{Z}$-module will do. For ease of notation, let $w_e := F(e)$ for each $e \in E(G)$. For a closed walk $W = \{v_j, v_{j+1}, \dots, v_k\}$, call $F(W) := \sum_{i=j}^{k-1} F(\{v_i, v_{i+1}\})$ the *weight* of the walk W. Note that $F(\overline{W_1}) = F(W_1)$ and $F(W_1 \circ W_2) = F(W_1) + F(W_2)$.

The reader can easily verify the following.

Proposition 1. *Let W_1 and W_2 be as above. If W_1 is a non-backtracking walk, and W_2 is a non-backtracking walk, then $\overline{W_1}$ is a non-backtracking walk, and $W_1 \circ W_2$ is a non-backtracking walk provided that $v_{k-1} \ne v_{k+1}$.*

Let $\mathcal{W}$ be a collection of closed non-backtracking walks in G. We say that an edge $e \in E(G)$ is *revealed* by $\mathcal{W}$ if there exist $W_1, W_2, \dots, W_\ell \in \mathcal{W}$ and non-zero integers $c_e, c_1, c_2, \dots, c_\ell$ so that $\sum_{i=1}^{\ell} c_i F(W_i) = c_e w_e$. In an analogous way, we say that a walk W is revealed by $\mathcal{W}$ when there exist $W_1, W_2, \dots, W_\ell \in \mathcal{W}$ and non-zero integers $c_W, c_1, c_2, \dots, c_\ell$ so that $\sum_{i=1}^{\ell} c_i F(W_i) = c_W F(W)$.

For a more linear-algebraic interpretation, we associate to each edge a vector of length $|E(G)|$ indexed by edges, where the only nonzero entry corresponds to the chosen edge, and this entry has the value of the weight of the edge. Similar to above, the vector corresponding to a walk is the sum of the edge vectors traversed. Then an edge (or walk) is revealed when the edge's vector can be written as $\mathbb{Q}$-linear combination of vectors corresponding to closed, non-backtracking walks. In fact, the linear algebraic definition yields the one above by clearing denominators and taking inner products with the all-ones vector.

For a vertex $v \in V(G)$, let $\mathcal{W}_v$ denote the set of all closed non-backtracking walks in G starting and ending at v. For a subset of vertices $S \subseteq V(G)$, we let $\mathcal{W}_S = \bigcup_{v \in S} \mathcal{W}_v$. For an edge $e \in E(G)$, we say e is *revealed by* S if e is revealed by $\mathcal{W}_S$. Define the *odometric set* of S, denoted $\mathcal{O}_S$, as the set of all edges revealed by S. If $O_S = E(G)$, we say that G is *odometric at* S. In the case that $S = \{v\}$ for some vertex v, we drop the set notation and say G is odometric at v. Finally, we say that the graph G is *odometric* if it is odometric at v for every vertex $v \in V(G)$. The main result of this paper is a complete characterization of odometric graphs. We remark that it is essential that we require the walks be both closed and non-backtracking, or else the above question would be trivial. To see why we require non-backtracking walks, let v be any vertex in a graph G, and let $e = \{u, v\}$ be any edge incident to v. $W = \{v, u, v\}$ is a closed (backtracking) walk anchored at v, and $F(W) = 2F(e)$. Evidently, one can reveal the entire neighborhood of v and (inductively) every edge in the connected component of G containing v. Hence if backtracking walks were permissible, a graph G would be odometric if and only if it were connected. Similarly, $W = \{v, u\} = e$ is a non-backtracking (non-closed) walk anchored at v with $F(W) = F(e)$, and again G is odometric if and only if it is connected.

Before moving to the characterization, we note a few subtleties in the above definitions. First, we note that any particular walk $W \in \mathcal{W}_S$ must have *one* vertex $v \in S$ for both its starting and ending vertex, which we term the *anchor* of the walk. However, we note that there could be walks $W_1, W_2 \in \mathcal{W}_S$ which are anchored at different vertices. Hence, in revealing some edge, one *is* able to use W_1 and W_2 despite their anchor vertices being different. For example, an edge could be revealed by using three closed walks from v, and two closed walks from w, provided $v, w \in S$.

Next, we address the question of using previously revealed edges in revealing later edges. An illustration is perhaps the simplest way to demonstrate. Suppose that u, v, w is a triangle in our graph, and we are trying to discover the odometric set for v. Suppose also that we can reveal the edge $e = \{v, u\}$ and the edge $f = \{u, w\}$. One would like to say that we can then reveal $g = \{w, v\}$ by getting the weight of the triangle $W = \{v, u, w, v\}$ as a closed walk, and then subtracting off the weights of e and f determine the weight of g.

While the definition of revealing an edge says nothing about using the weights of previously revealed edges, our next proposition shows that using such information does not increase the size of an odometric set.