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On the signed total Roman domination and domatic numbers of graphs

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ABSTRACT

A *signed total Roman dominating function* on a graph G is a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u\in N(v)} f(u) \geq 1$ for every vertex $v \in V(G)$, where $N(v)$ is the neighborhood of v, and every vertex $u \in V(G)$ for which $f(u) = -1$ is adjacent to at least one vertex w for which $f(w) = 2$. The *signed total Roman domination number* of a graph G, denoted by $\gamma_{stR}(G)$, equals the minimum weight of a signed total Roman dominating function. A set $\{f_1, f_2, \dots, f_d\}$ of distinct signed total Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed total Roman dominating family* (of functions) on G. The maximum number of functions in a signed total Roman dominating family on G is the *signed total Roman domatic number* of G, denoted by $d_{stR}(G)$. In this paper we continue the investigation of the signed total Roman domination number, and we initiate the study of the signed total Roman domatic number in graphs. We present sharp bounds for $\gamma_{stR}(G)$ and $d_{stR}(G)$. In addition, we determine the total signed Roman domatic number of some graphs.

1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [3]. Specifically, let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A graph G is *regular* or *r-regular* if $d(v) = r$ for each vertex v of G. Let S be a set of vertices, and let $u \in S$. We say that v is a *private neighbor* of u (with respect to S) if $N[u] \cap S = \{u\}$. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n, $K_{p,q}$ for the *complete bipartite graph* with partite sets X and Y, where $|X| = p$ and $|Y| = q$, C_n for the *cycle* of length n, and P_n for the path of order n.

In this paper we continue the study of Roman dominating functions in graphs and digraphs. Following the ideas in [1], we defined in [7] the *signed total Roman dominating function* (STRDF) on a graph G as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $f(N(v)) = \sum_{u\in N(v)} f(u) \geq 1$ for each $v \in V(G)$, and such that every vertex $u \in V(G)$ for which $f(u) = -1$ is adjacent to at least one vertex w for which $f(w) = 2$. The *weight* of an STRDF f is the value $\omega(f) = \sum_{v\in V(G)} f(v)$. The *signed total Roman domination number* of a graph G, denoted by $\gamma_{stR}(G)$, equals the minimum weight of an STRDF on G. We note that this parameter is only defined for graphs without isolated vertices. Thus we assume throughout this paper that $\delta(G) \geq 1$. A $\gamma_{stR}(G)$-*function* is a signed total Roman dominating function of G with weight $\gamma_{stR}(G)$. A signed total Roman dominating function $f : V(G) \longrightarrow \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$, where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = -1, 1, 2$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.

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A concept dual in a certain sense to the domination number is the domatic number, introduced by Cockayne and Hedetniemi [2]. They have defined the domatic number $d(G)$ of a graph G by means of sets. A partition of $V(G)$, all of whose classes are dominating sets in G, is called a domatic partition. The maximum number of classes of a domatic partition of G is the domatic number $d(G)$ of G. But Rall has defined a variant of the domatic number of G, namely the fractional domatic number of G, using functions on $V(G)$. (This was mentioned by Slater and Trees in [6].) Analogous to the fractional domatic number we may define the signed total Roman domatic number.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \le 1$ for each $v \in V(G)$, is called a *signed total Roman dominating family* (of functions) on G. The maximum number of functions in a signed total Roman dominating family (STRD family) on G is the *signed total Roman domatic number* of G, denoted by $d_{stR}(G)$. The signed total Roman domatic number is well-defined and $d_{stR}(G) \ge 1$ for all graphs G with $\delta(G) \ge 1$, since the set consisting of any STRDF forms an STRD family on G. In [5], Sheikholeslami and Volkmann investigated the signed Roman domatic number in graphs.

Our purpose in this paper is to continue the investigations of the signed total Roman domination number and to initiate the study of the signed total Roman domatic number in graphs. First, we characterize the graphs G with $\gamma_{stR}(G) = n(G)$. Second, we derive basic properties and bounds for the signed total Roman domatic number of a graph. In particular, we obtain different upper bounds on the sum $\gamma_{stR}(G) + d_{stR}(G)$. In addition, we determine the signed total Roman domatic number of some classes of graphs. Finally, we derive the Nordhaus–Gaddum type result

$$d_{stR}(G) + d_{stR}(\overline{G}) \le n - 1,$$

with equality if and only if $G = C_4$ and $\overline{G} = K_2 \cup K_2$.

We make use of the following known results in this paper.

Proposition A ([7]). *If G is a δ-regular graph of order n with $\delta \ge 1$, then*

$$\gamma_{stR}(G) \ge \left\lceil \frac{n}{\delta} \right\rceil.$$

Proposition B ([7]). *If C_n is a cycle of order $n \ge 3$, then $\gamma_{stR}(C_n) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma_{stR}(C_n) = (n+3)/2$ when $n \equiv 1, 3 \pmod 4$ and $\gamma_{stR}(C_n) = (n+6)/2$ when $n \equiv 2 \pmod 4$.*

Proposition C ([7]). *If P_n is a path of order $n \ge 3$, then $\gamma_{stR}(P_n) = n/2$ when $n \equiv 0 \pmod 4$, and $\gamma_{stR}(P_n) = \lceil (n+3)/2 \rceil$ otherwise.*

Proposition D ([7]). *For $p \ge 1$, $\gamma_{stR}(K_{p,p}) = 2$, unless $p = 3$ in which case $\gamma_{stR}(K_{3,3}) = 4$.*

Proposition E ([7]). *If $n \ge 3$ is an integer, then $\gamma_{stR}(K_n) = 3$.*

Proposition F ([7]). *Let G be a graph of order n. If $\delta(G) \ge 1$, then $\gamma_{stR}(G) \le n$ and if $\delta(G) \ge 3$, then $\gamma_{stR}(G) \le n - 1$.*

2. Signed total Roman domination number

First we improve Proposition A for 3-regular graphs considerably.

Theorem 1. *If G is a 3-regular graph of order n, then*

$$\gamma_{stR}(G) \ge \frac{2n}{3},$$

with equality if and only if $|V_{-1}| = |V_1| = |V_2|$ for every γ_{stR}-function $f = (V_{-1}, V_1, V_2)$ on G.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stR}(G)$-function. If $V_{-1} = \emptyset$, then $\gamma_{stR}(G) = n \ge (2n)/3$. Thus let now $V_{-1} = \{u_1, u_2, \ldots, u_t\} \ne \emptyset$. As every vertex of V_{-1} is adjacent to at least one vertex in V_2, we choose to every vertex $u_i \in V_{-1}$ exactly one vertex $u_i' \in V_2$ for $1 \le i \le t$. Since G is 3-regular, the condition $f(N(v)) \ge 1$ for each vertex v, shows that no vertex of G has two neighbors in V_{-1}. If $V_2' = \{u_1', u_2', \ldots, u_t'\}$, then we thus deduce that $|V_2'| = |V_{-1}|$. Furthermore, every vertex of V_{-1} has a private neighbor in $(V_2 - V_2') \cup V_1$ and therefore $|(V_2 - V_2') \cup V_1| \ge |V_{-1}|$. Combining these observations, we obtain

$$\begin{aligned} 3\omega(f) &= 3(2|V_2| + |V_1| - |V_{-1}|) \ge 3(|V_2| + |V_1|) \\ &= 2|V_2| + 2|V_1| + |V_2'| + |(V_2 - V_2') \cup V_1| \\ &\ge 2|V_2| + 2|V_1| + 2|V_{-1}| = 2n \end{aligned}$$

and so $\gamma_{stR}(G) = \omega(f) \ge (2n)/3$. If $\gamma_{stR}(G) = \frac{2n}{3}$, then the two inequalities occurring in the inequality chain above become equalities, and this implies $|V_{-1}| = |V_1| = |V_2|$. Conversely, if $|V_{-1}| = |V_1| = |V_2|$, then $\gamma_{stR}(G) = 2|V_2| + |V_1| - |V_{-1}| = |V_2| + |V_1| = 2n/3$. □

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