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Extremal values and bounds for the zero forcing number

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ABSTRACT

A set Z of vertices of a graph G is a zero forcing set of G if iteratively adding to Z vertices from $V(G) \setminus Z$ that are the unique neighbor in $V(G) \setminus Z$ of some vertex in Z, results in the entire vertex set $V(G)$ of G. The zero forcing number $Z(G)$ of G is the minimum cardinality of a zero forcing set of G.

Amos et al. (2015) proved $Z(G) \leq ((\Delta - 2)n + 2)/(\Delta - 1)$ for a connected graph G of order n and maximum degree $\Delta \geq 2$. Verifying their conjecture, we show that C_n, K_n, and $K_{\Delta,\Delta}$ are the only extremal graphs for this inequality. Confirming a conjecture of Davila and Kenter [5], we show that $Z(G) \geq 2\delta - 2$ for every triangle-free graph G of minimum degree $\delta \geq 2$. It is known that $Z(G) \geq P(G)$ for every graph G where $P(G)$ is the minimum number of induced paths in G whose vertex sets partition $V(G)$. We study the class of graphs G for which every induced subgraph H of G satisfies $Z(H) = P(H)$.

1. Introduction

We consider graphs that are finite, simple, and undirected, and use standard terminology.

Let G be a graph. A set Z of vertices of G is a *zero forcing set* of G if every proper subset $\bar{Z}$ of the vertex set $V(G)$ of G with $Z \subseteq \bar{Z}$ contains a vertex that has exactly one neighbor in $V(G) \setminus \bar{Z}$. Equivalently, Z is zero forcing set of G if there is a linear ordering $u_1, \ldots, u_k$ of the vertices in $V(G) \setminus Z$ such that for every index $i \in [k]$, there is a vertex v_i in $Z \cup \{u_1, \ldots, u_{i-1}\}$ such that u_i is the unique neighbor of v_i in $\{u_i, \ldots, u_k\}$. In this case we say that v_i *forces* u_i and denote this by $v_i \to u_i$. The sequence $v_1 \to u_1, v_2 \to u_2, \ldots, v_k \to u_k$ is called a *forcing sequence* for Z. Note that a forcing sequence specifies a linear order in which the vertices in $V(G) \setminus Z$ can be forced one after the other, and that neither the choice of the v_i nor of the forcing sequence is necessarily unique. The *zero forcing number* $Z(G)$ of G is the minimum cardinality of a zero forcing set of G.

This parameter was introduced independently by the AIM Minimum Rank—Special Graphs Work Group [1] with an algebraic motivation in mind, and by Burgarth and Giovannetti [3] with a physical motivation in mind. It has already been studied in a number of papers, for instance [2,4–8,10–13].

Our contributions are as follows. In Section 2 we confirm a conjecture of Amos et al. [2] concerning the extremal graphs for some upper bound on the zero forcing number. In Section 3 we prove a lower bound on the zero forcing number of triangle-free graphs, which was conjectured by Davila and Kenter [5]. Finally, in Section 4, we extend a result of Row [11] concerning the graphs for which the zero forcing number equals a path cover number.

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2. Extremal graphs for two upper bounds

In [2] Amos et al. prove the following upper bounds on the zero forcing number.

Theorem 1 (*Amos et al. [2]*). *Let G be a graph of order n, maximum degree Δ, and minimum degree at least 1.*

(i) $Z(G) \leq \frac{\Delta n}{\Delta+1}$.
(ii) *If G is connected and Δ is at least 2, then* $Z(G) \leq \frac{(\Delta-2)n+2}{\Delta-1}$.

They conjecture (cf. Conjecture 6.1 in [2]) that the only extremal graphs for Theorem 1(ii) are the cycle C_n, the complete graph K_n, and the balanced complete bipartite graph $K_{\Delta,\Delta}$.

Our first goal is to prove this conjecture.

The following is a variant of Lemma 4.1 in [2].

Lemma 2. *Let G be a connected graph of order n at least 2. If Z_0 is a zero forcing set of G such that $|Z_0| < n$ and $G - Z_0$ is connected, then there is a zero forcing set Z of G such that $Z \subseteq Z_0$, $G - Z$ is connected, and every vertex in Z has a neighbor in $V(G) \setminus Z$.*

Proof. Choose a zero forcing set Z of G such that $Z \subseteq Z_0$, $G - Z$ is connected, and, subject to these conditions, Z is minimal with respect to inclusion. For a contradiction, we assume that some vertex in Z has no neighbor in $V(G) \setminus Z$. Since G is connected and has order at least 2, there is a path uvw such that $u, v \in Z$, $w \notin Z$, and u has no neighbor in $V(G) \setminus Z$. If $Z' = Z \setminus \{v\}$, then $G - Z'$ is connected. Furthermore, since v is the unique neighbor of u in $V(G) \setminus Z'$, the set Z' is a zero forcing set of G, which is a contradiction. □

We proceed to the proof of the conjecture of Amos et al. concerning the extremal graphs for Theorem 1(ii).

Theorem 3. *If G is a connected graph of order n and maximum degree Δ at least 2, then*

$$Z(G) \leq \frac{(\Delta-2)n+2}{\Delta-1} \tag{1}$$

with equality if and only if G is either C_n, or K_n, or $K_{\Delta,\Delta}$.

Proof. Let G be a connected graph of order n and maximum degree Δ.

Let Z_0 be an arbitrary zero forcing set of G such that $|Z_0| < n$ and $G - Z_0$ is connected. Note that such a set exists; in fact, every set of order $n-1$ has these properties. Let Z be as in Lemma 2. Let m be the number of edges between Z and $V(G) \setminus Z$, and let m' be the number of edges of $G - Z$. By Lemma 2,

$$m \geq |Z|. \tag{2}$$

Since $G - Z$ is connected

$$m' \geq n - |Z| - 1. \tag{3}$$

Since the maximum degree is Δ,

$$m \leq \Delta(n - |Z|) - 2m'. \tag{4}$$

This implies

$$|Z| \overset{(2)}{\leq} m \overset{(4)}{\leq} \Delta(n-|Z|) - 2m' \overset{(3)}{\leq} \Delta(n-|Z|) - 2(n-|Z|-1), \tag{5}$$

which is equivalent to (1).

We proceed to the characterization of the extremal graphs.

If $G = C_n$, then $Z(G) = 2$ and $\frac{(\Delta-2)n+2}{\Delta-1} = \frac{(2-2)n+2}{2-1} = 2$.

If $G = K_n$, then $Z(G) = n-1$ and $\frac{(\Delta-2)n+2}{\Delta-1} = \frac{(n-3)n+2}{n-2} = n-1$.

If $G = K_{\Delta,\Delta}$, then $Z(G) = 2\Delta - 2$ and $\frac{(\Delta-2)n+2}{\Delta-1} = \frac{(\Delta-2)2\Delta+2}{\Delta-1} = 2\Delta - 2$.

Therefore, if G is either C_n, or K_n, or $K_{\Delta,\Delta}$, then (1) holds with equality.

Now let (1) hold with equality. It remains to show that G is either C_n, or K_n, or $K_{\Delta,\Delta}$. In view of (5), equality in (1) implies that for every choice of Z_0 and every choice of Z, the inequalities (2), (3), and (4) hold with equality. Equality in (2) implies that

$$\text{every vertex in } Z \text{ has exactly one neighbor in } V(G) \setminus Z. \tag{6}$$

Equality in (3) implies that

$$G - Z \text{ is a tree.} \tag{7}$$