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Note

The Territorial Raider game and graph derangements

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ABSTRACT

A derangement of a graph $G = (V, E)$ is an injective function $f : V \to V$ such that for all $v \in V, f(v) \neq v$ and $(v, f(v)) \in E$. Not all graphs admit a derangement and previous results have characterized graphs with derangements using neighborhood conditions for subsets of V. We establish an alternative criterion for the existence of derangements on a graph. We analyze strict Nash equilibria of the biologically motivated Territorial Raider game, a multi-player competition for resources in a spatially structured population based on animal raiding and defending behavior. We find that a graph G admits a derangement if and only if there is a strict Nash equilibrium of the Territorial Raider game on G.

1. Introduction

A set derangement is a permutation of a set's elements with no fixed points [9]. Similarly, a graph derangement is a permutation of the vertices of a graph which has no fixed points, with additional limitations imposed by the structure of the graph. In this paper, any graph referred to is assumed to be simple, finite, undirected, and connected. Under this definition, a graph derangement is an injective function mapping all vertices of a graph to adjacent vertices [6]. While derangements exist for all sets containing more than one element, the existence of a derangement of a graph depends on its structure. For instance, a three vertex path graph permits no derangements [6].

Tutte (1953) introduced the idea of the Q-factor of an unoriented graph [11, p. 930]. A Q-factor is a spanning subgraph which consists of 1-regular components (vertex pairs), and 2-regular components (cycles). A graph will have a Q-factor if and only if that graph has a derangement. This can be seen by writing a derangement in cycle notation, as each cycle indicates a 1 or 2-regular subgraph of the original graph [6].

A finite graph G with vertex set V admits a derangement, or equivalently, has a Q-factor, if and only if, for any subset $W \subseteq V$, $|N(W)| \geq |W|$, where $N(W)$ is the set of all vertices adjacent to a vertex in W [6,11].

In this paper, we provide a new criterion for the existence of graph derangements. We adapt the Territorial Raider game (see for example [2–5]) and establish a one-to-one correspondence between a derangement of a graph and a strict Nash equilibrium of the Territorial Raider game. This will prove that a simple, finite, undirected, connected graph G admits a derangement if and only if a Territorial Raider game played on G has a strict Nash equilibrium. This game-theoretical approach allows graphs to be analyzed through the implementation of multi-agent machine-learning algorithms such as Exp3 [1] which can potentially determine Nash equilibria, and thus the existence of derangements [10].

We note that [6] introduced graph derangements using the example of cockroaches skittering on a 5×5 checkerboard tiled floor. From a graph theoretical perspective, the floor is a bipartite graph of odd cardinality and thus is has no graph

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derangement. From the game-theoretical perspective, if the tiles are colored in a usual checkerboard pattern with the corners colored black, then there are 13 black squares and 12 white squares, and so the cockroaches on the 13 black squares cannot all move to different white squares. Consequently, there is no strict Nash equilibrium and whatever the roaches do, at least one roach will have regrets.

2. Definitions and preliminaries

Definition 2.1. Let $G = (V, E)$ be a graph. A *graph permutation* is a bijection such that $v \in V, f(v) = v$ or $(v, f(v)) \in E$. A *graph derangement* of G is a graph permutation $f : V \to V$ such that for all $v \in V$, $(v, f(v)) \in E$ (i.e. $f(v) \neq v$).

The *Territorial Raider game* is played on a graph $G = (V, E)$. Every vertex $v \in V$ is occupied by a player I_v and the vertex v is called the *home vertex* of I_v. All vertices contain one unit of resources. All players must simultaneously choose whether to raid a neighboring vertex or stay home to defend against potential raiders. The object of the game is to maximize the resources obtained.

We are interested in determining the *strict Nash equilibria*, which are sets of the strategies of all players such that any player will reduce their payoff by unilaterally changing their strategy ([7], p. 11–12). We note that strict Nash equilibria must consist of pure strategies, see for example [8]. Consequently, we will only consider pure strategies.

Formally, a *strategy* for player I_v is a choice of a vertex $w \in V$ such that $w = v$ or $(v, w) \in E$. An *admissible function* of G is a function $f : V \to V$ such that for all $v \in V, f(v) = v$ or $(v, f(v)) \in E$. We can see that there is a one-to-one correspondence between strategy sets for the players and admissible functions of G. We will use f^{-1} to denote the inverse of f when f is bijective, and the preimage of f otherwise.

When all individuals move according to their strategy, they receive payoffs based on their position as well as the positions of their opponents. By staying home, a player guarantees its claim to a portion $h \in [0, 1]$ of its own initial unit of resources. The value h is a fixed parameter chosen before the game begins. The remaining $1-h$ resources are then split equally between all occupants of a vertex. If a player raids, it loses all of its own initial unit of resources to raiders unless no other player raids its home vertex, in which case it keeps all of its resources coming from the home vertex. Given the game rules, a player's payoff will be in the range of $[0, 2]$.

Specifically, if $f(v) = v$ (a player I_v chooses to defend), that player will receive

$$P_v(f) = h + \frac{(1-h)}{|f^{-1}(v)|} \tag{1}$$

where $|f^{-1}(v)|$ denotes the cardinality of the preimage of f, and thus the total number of players at vertex v. If $f(v) = v' \neq v$ (a player I_v chooses to raid node v'), then the payoff is

$$P_v(f) = \begin{cases} 1 + \dfrac{1-h}{|f^{-1}(v')|} & \text{if } f^{-1}(v) = \emptyset, f(v') = v' \text{ (no player raids } v \text{ and } I_{v'} \text{ defends)} \\ 1 + \dfrac{1}{|f^{-1}(v')|} & \text{if } f^{-1}(v) = \emptyset, f(v') \neq v' \text{ (no player raids } v \text{ and } I_{v'} \text{ raids)} \\ \dfrac{1-h}{|f^{-1}(v')|} & \text{if } f^{-1}(v) \neq \emptyset, f(v') = v' \text{ (some player raids } v \text{ and } I_{v'} \text{ defends)} \\ \dfrac{1}{|f^{-1}(v')|} & \text{if } f^{-1}(v) \neq \emptyset, f(v') \neq v' \text{ (some player raids } v \text{ and } I_{v'} \text{ raids).} \end{cases} \tag{2}$$

We note that in order for a strict Nash equilibrium to exist, we must have $h < 1$. Indeed, for a contradiction, assume $h = 1$ and that the Nash equilibrium is generated by $f : V \to V$. If $f(v) = v$ for all $v \in V$, then any individual can raid a neighbor and its payoff stays the same. If there is a $v \in V$ such that $f(v) \neq v$, then individual I_v can stay home, receive the payoff of 1, and thus not reduce its payoff.

The main result of this paper is the following theorem.

Theorem 2.1. *A simple, finite, undirected, and connected graph G admits a derangement if and only if a Territorial Raider game played on G, with* $h \in [0, 1)$*, has a strict Nash equilibrium strategy set.*

In fact, our argument will show that a graph permutation $f : V \to V$ is a graph derangement if and only if the corresponding pure strategy set is a strict Nash equilibrium.

3. Proof of Theorem 2.1

We will first show that any derangement generates a strict Nash equilibrium (Proposition 3.1). Then, we will show that any strict Nash equilibrium must be generated by a derangement (Theorem 3.3). Theorem 2.1 follows directly from these two results.

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