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Proximity, remoteness and distance eigenvalues of a graph

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ABSTRACT

Proximity π and remoteness ρ are respectively the minimum and the maximum, over the vertices of a connected graph, of the average distance from a vertex to all others. The distance spectral radius ∂_1 of a connected graph is the largest eigenvalue of its distance matrix. In the present paper, we are interested in a comparison between the proximity and the remoteness of a simple connected graph on the one hand and its distance eigenvalues on the other hand. We prove, among other results, lower and upper bounds on the distance spectral radius using proximity and remoteness, and lower bounds on $\partial_1-\pi$ and on $\partial_1-\rho$. In addition, several conjectures, obtained with the help of the system AutoGraphiX, are formulated.

1. Introduction

Let $G=(V,E)$ denote a simple and connected graph, with vertex set V and edge set E, containing $n=|V|$ vertices and $m=|E|$ edges. All the graphs considered in the present paper are finite, simple and connected. The distance between two vertices u and v in G, denoted by $d(u,v)$, is the length of a shortest path between u and v. The average distance between all pairs of vertices in G is denoted by $\overline{\ell}$. The eccentricity $e(v)$ of a vertex v in G is the largest distance from v to another vertex of G. The minimum eccentricity in G, denoted by r, is the *radius* of G. The maximum eccentricity of G, denoted by D, is the *diameter* of G. The *average eccentricity* of G is denoted ecc. That is

$$r=\min_{v\in V} e(v), \qquad D=\max_{v\in V} e(v) \quad \text{and} \quad ecc=\frac{1}{n}\sum_{v\in V} e(v).$$

The *proximity* π of G is the minimum average distance from a vertex of G to all others. Similarly, the *remoteness* ρ of G is the maximum average distance from a vertex to all others. The two last concepts were introduced in [1,4]. They are close to the concept of transmission $t(v)$ of a vertex v, which is the sum of the distances from v to all others. That is, if we denote $\tilde{t}(v)$ as the average distance from a vertex v to all other vertices in G, we have

$$\pi=\min_{v\in V}\tilde{t}(v)=\min_{v\in V}\frac{t(v)}{n-1} \quad \text{and} \quad \rho=\max_{v\in V}\tilde{t}(v)=\max_{v\in V}\frac{t(v)}{n-1}.$$

The transmission of a vertex is also known as the *distance* of a vertex [16] and the minimum distance (transmission) of a vertex is studied in [33]. A notion very close to the average distance from a vertex is the *vertex deviation* introduced by Zelinka [44] as

$$m_1(v)=\frac{1}{n}\sum_{u\in V} d(u,v)=\frac{t(v)}{n}.$$

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The vector composed of the vertex transmissions in a graph was first introduced by Harary [22] in 1959, under the name *status* of a graph, as a measure of the "weights" of individuals in social networks. The same vector was called *the distance degree sequence* by Bloom, Kennedy and Quintas [10]. It was used to tackle the problem of graph isomorphism. Randić [34] conjectured that two graphs are isomorphic if and only if they have the same distance degree sequence. The conjecture was refuted by several authors such as Slater [39], Buckley and Harary [11], and Entringer, Jackson and Snyder [16]. The transmission of a graph was also introduced by Sabidussi [36] in 1966 as a measure of *centrality* in social networks. The notion of centrality is widely used in different branches of science (see for example [27] and the references therein) such as transportation-network theory, communication network theory, electrical circuits theory, psychology, sociology, geography, game theory and computer science. Notions closely related to that of the distance from a vertex are those of *a center* and *a centroid* already introduced by Jordan [26] in 1869. For an overview of mathematical properties of these two concepts see the survey [40], as well as the references therein. In 1964, Hakimi [21] used for the first time the sum of distances in solving facility location problems. In fact, Hakimi [21] considered two problems, subsequently considered in many works: the first problem was to determine a vertex $u \in V$ so as to minimize $\max_{v \in V}\{d(u, v) : u \in V\}$, *i.e.*, the center of a graph; and the second problem is to determine a vertex $u \in V$ so as to minimize the sum of distances from u, *i.e.*, the centroid. Interpretations of these problems can be found, for instance, in [18]. In view of the interest of the transmission vector in different domains of science, it is natural to study the properties of its extremal values themselves, and among the set of graph parameters. A direct study of the minimum and the maximum values of the transmission is not adequate since they do not have the same order of magnitude as the majority of the distance graph parameters. Proximity and remoteness appear to be more convenient than minimum and maximum transmissions in comparisons with other metric invariants, such as the diameter, radius, average eccentricity and average distance, as they have the same order of magnitude when viewed as functions of the order n of G. Indeed, it follows from the definitions that

$$\pi \le r \le ecc \le D, \qquad \pi \le \overline{\ell} \le \rho \le D,$$

where $\overline{\ell}$ denotes the average distance, *i.e.*,

$$\overline{\ell} = \frac{1}{n(n-1)} \sum_{(u,v) \in V \times V} d(u, v) = \frac{1}{n(n-1)} \sum_{v \in V} t(v).$$

Since their introduction in [1,4] and the first studies of their properties [6,8], proximity and remoteness attracted the attention of several authors. In [38], Sedlar et al. proved a series of AutoGraphiX (a software devoted to conjecture-making in graph theory, see [1,2,12,13]) conjectures in two of which remoteness ρ is involved. Those results are gathered in the next theorem. First, recall the following definitions. The (*vertex*) *connectivity* ν of a connected graph G is the minimum number of vertices whose removal disconnects G or reduces it to a single vertex. The *algebraic connectivity* a of a graph G is the second smallest eigenvalue of its Laplacian $L = \mathrm{Diag} - A$, where Diag is the diagonal square matrix indexed by the vertices of G whose diagonal entries are the degrees in G, and A is the adjacency matrix of G.

Theorem 1.1 (*[38]*). *Let G be a connected graph on $n \ge 2$ vertices with vertex connectivity ν, algebraic connectivity a and remoteness ρ. Then*

$$\nu \cdot \rho \le n - 1$$

with equality if and only if G is the complete graph K_n; and

$$a \cdot \rho \le n$$

with equality if and only if G is the complete graph K_n. Moreover, if G is not complete, then

$$a \cdot \rho \le n - 1 - \frac{1}{n-1}$$

with equality if and only if $G \cong K_n - M$, where M is any non empty set of disjoint edges.

Ma, Wu and Zhang [31] proved an AutoGraphiX conjecture that consists of an upper bound on $ecc - \pi$ together with the corresponding extremal graphs.

Theorem 1.2 (*[31]*). *For a connected graph G on $n \ge 3$ vertices,*

$$ecc - \pi \le \begin{cases} \dfrac{(3n+1)(n-1)}{4n} - \dfrac{n+1}{4} & \text{if } n \text{ is odd,} \\[2ex] \dfrac{n-1}{2} - \dfrac{n}{4(n-1)} & \text{if } n \text{ is even,} \end{cases}$$

with equality if and only if G is the path P_n.

Sedlar [37] studied three AutoGraphiX conjectures involving proximity and remoteness. She solved the conjecture stated in the following theorem.