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On the path separation number of graphs

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ABSTRACT

A *path separator* of a graph G is a set of paths $\mathcal{P} = \{P_1, \dots, P_t\}$ such that for every pair of edges $e, f \in E(G)$, there exist paths $P_e, P_f \in \mathcal{P}$ such that $e \in E(P_e), f \notin E(P_e), e \notin E(P_f)$ and $f \in E(P_f)$. The *path separation number* of G, denoted $\operatorname{psn}(G)$, is the smallest number of paths in a path separator. We shall estimate the path separation number of several graph families – including complete graphs, random graph, the hypercube – and discuss general graphs as well.

1. Introduction

Separation of combinatorial objects is a well-studied question in mathematics and engineering, dating back to at least Rényi [18]. In fact the notion goes by many terms: identification or, in engineering, *testing* are also used for the same idea [4,6,9,11,12,19].

Let $\mathcal{H} = (X, E)$ be a hypergraph with ground set X and edge set E. We say that $L \subset E$ is a *weak separating system* if for all $x, y \in X,\ x \neq y$ there exists an $A \in L$ such that either $x \in A$ or $y \in A$, but $\{x, y\} \not\subset A$. Similarly, L is a *strong (or complete) separating system* if for all $x, y \in X,\ x \neq y$ there exist $A_x, A_y \in L$ such $x \in A_x$ and $y \in A_y$, but $x \notin A_y$ and $y \notin A_x$, as defined by Dickson [5]. Observe that any strong separating system is also a weak separating system. In several applications X is just the vertices or edges of a certain graph G, while E can be a set of closed neighborhoods, cycles, closed walks, paths, etc. of G, see e.g. [8,6,12,19].

In this paper we consider strong separation of the edges of graphs by paths. Since we deal with strong separation in this paper, we will just use the term "separating system" or "separator" when referring to a strong separating system. Let G be a graph with at least two edges. A *path separator* of G is a set of paths $\mathcal{P} = \{P_1, \dots, P_t\}$ such that for every pair of distinct edges $e, f \in E(G)$, there exist paths $P_e, P_f \in \mathcal{S}$ such that $e \in E(P_e)$ and $f \in E(P_f)$ but $e \notin E(P_f)$ and $f \notin E(P_e)$. The *path separation number* of G, denoted $\operatorname{psn}(G)$, is the smallest number of paths in a path separator. If G has exactly one edge then we say that $\operatorname{psn}(G) = 1$ and if G is empty then we say that $\operatorname{psn}(G) = 0$.

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Rényi [18] conjectured that $O(n)$ paths suffice for the weak separation in any graph with n vertices. This problem is still unsolved, although Falgas-Ravry, Kittipassorn, Korándi, Letzer and Narayanan [8] recently made some progress for proving it for trees and certain random graphs. We propose the stronger Conjecture 11: $O(n)$ paths are sufficient even for strong separation.

In this paper we prove this conjecture for complete graphs (Theorem 4), forests (Theorem 5), higher dimensional cubes (Theorem 8), and not too sparse random graphs (Theorem 9). It is somehow surprising since generally strong separation may need many more paths than weak separation, as we remark following Theorem 8.

Denote $H_2(x)$ to be the *binary entropy function*, i.e. $H_2(x) = -x\log_2 x - (1-x)\log_2(1-x)$, where $x \in (0, 1)$. Denote K_n to be the complete graph and P_n to be the path on n vertices. The parameters $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G, respectively.

Fact 1 follows from the fact that the edge set itself is a path separator if there are at least 2 edges and $\mathrm{psn}(G) = m$ if $m = 1$ or $m = 0$.

Fact 1. *Let G be a graph with m edges. Then* $\mathrm{psn}(G) \leq m$.

Because of Fact 2, we will always assume that the graph G that we are working with is connected.

Fact 2. *If G is a graph that is the vertex-disjoint union of graphs* G_1 *and* G_2 *then* $\mathrm{psn}(G) = \mathrm{psn}(G_1) + \mathrm{psn}(G_2)$.

When G is a forest we determine $\mathrm{psn}(G)$ in Theorem 5, otherwise Theorem 3 estimates it. Note that the proof of the lower bound in Theorem 3 does not use the structure of paths, only that a path has at most $n-1$ edges.

Theorem 3. *Let G be a graph on* $n \geq 4$ *vertices and* $m \geq 2(n-1)$ *edges, then*

$$\frac{m\ln m}{n\ln(en/2)} < \frac{\log_2 m}{H_2\left((n-1)/m\right)} \leq \mathrm{psn}(G) \leq 4n\lceil\log_2\lceil m/n\rceil\rceil + 2n < 5n\log_2 n.$$

Theorem 4 establishes that the path separation number of the complete graph is at most $2n+4$ and Theorem 3 implies that it is at least $(1-o(1))n$. Of course, the bound $2n+4$ is not sharp even for $n = 5$ or 6, since by Fact 1 we have that $\mathrm{psn}(K_n) \leq n(n-1)/2 < 2n+4$ in these cases.

Theorem 4. *For* $n \geq 10$ *we have* $\mathrm{psn}(K_n) \leq 4\lceil n/2\rceil + 2 \leq 2n+4$.

Theorem 5 gives an explicit formula for the path separation number of a forest F depending only on the degree sequence and the number of connected components of F that are, themselves, paths. A *path-component* of a graph is a connected component that is a path.

Theorem 5. *Let F be a forest with* v_1 *vertices of degree* 1, v_2 *vertices of degree* 2 *and p path-components. Then* $\mathrm{psn}(F) = v_1 + v_2 - p$.

Corollary 6. *The smallest path separation number for a tree T on n vertices is* $\lceil n/2\rceil + 1$. *This is achieved with equality if and only if* (a) *n is even and all the degrees of T are either* 1 *or* 3 *or* (b) *n is odd, T has one vertex of degree either* 2 *or* 4 *and all other vertices have degree either* 1 *or* 3.

Corollary 7. *If G is a tree with n vertices then* $\mathrm{psn}(G) = n-1$ *if and only if G is a subdivided star.*

Theorem 8 considers the graph of the d-dimensional hypercube Q_d, whose path separation number shows different behavior from our previous results.

Theorem 8. *For* $d \geq 2$, *let* Q_d *denote the d-dimensional hypercube. Then* $\frac{d^2}{4\ln d} \leq \mathrm{psn}(Q_d) \leq 3d^2 + d - 4$.

Theorem 8 also demonstrates the difference between weak and strong separation: Honkala, Karpovsky and Litsyn proved in [12] that essentially $d + \log_2 d$ cycles are necessary and sufficient for a weak separation of the edges of the hypercube, which easily translates to a weak path separator having essentially at most $2(d + \log_2 d)$ paths, that is, much less than what is required in any strong separating system.

In Theorem 9 we address the Erdős–Rényi random graph in which each pair of vertices is, independently, chosen to be an edge with probability p. We say that a sequence of random events occurs *with high probability* if the probability of the events approaches 1 as $n \to \infty$.

Theorem 9. *Let* $p = p(n) > 1000\log n/n$ *and* $s = 4\log n/\log(pn/\log n)$. *Then* $\mathrm{psn}(G(n,p)) = O(\mathrm{psn})$ *with high probability.*

In particular, for $\alpha > 0$ *and* $p = p(n) > n^{\alpha-1}$ *this gives* $\mathrm{psn}(G(n,p)) = \Theta(pn)$ *with high probability and for* $p = p(n) > 10\log n/n$ *it yields that* $\mathrm{psn}(G(n,p)) = O(pn\log n)$, *with high probability.*