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Word-representability of triangulations of grid-covered cylinder graphs

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ABSTRACT

A graph $G = (V, E)$ is word-representable if there exists a word w over the alphabet V such that letters x and y, $x \neq y$, alternate in w if and only if $(x, y) \in E$. Halldórsson, Kitaev and Pyatkin have shown that a graph is word-representable if and only if it admits a so-called semi-transitive orientation. A corollary of this result is that any 3-colorable graph is word-representable.

Akrobotu, Kitaev and Masárová have shown that a triangulation of a grid graph is word-representable if and only if it is 3-colorable. This result does not hold for triangulations of grid-covered cylinder graphs; indeed, there are such word-representable graphs with chromatic number 4. In this paper we show that word-representability of triangulations of grid-covered cylinder graphs with three sectors (resp., more than three sectors) is characterized by avoiding a certain set of six minimal induced subgraphs (resp., wheel graphs W_5 and W_7).

1. Introduction

Let $G = (V, E)$ be a simple (i.e. without loops and multiple edges) undirected graph with the vertex set V and the edge set E. We say that G is *word-representable* if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if $(x, y) \in E$ for any $x \neq y$. By definition, each letter in V must appear in w.

The notion of word-representable graphs has its roots in algebra, where a prototype of these graphs was used by Kitaev and Seif to study the growth of the free spectrum of the well known *Perkins semigroup* [11].

A number of results on word-representable graphs are available in the literature [1–3,5,7,6,8,10,12,13]. In particular, Halldórsson, Kitaev and Pyatkin [8] have shown that a graph is word-representable if and only if it admits a *semi-transitive orientation* (to be defined in Section 2), which, among other important corollaries, implies that all 3-colorable graphs are word-representable. We refer to [9] for the state of the art in the theory of word-representable graphs.

Most relevant to our paper are [1,2,5], where *triangulations* and *subdivisions* of certain graphs are studied with respect to word-representability. In particular, Akrobotu, Kitaev and Masárová [1] proved that any triangulation of the graph G associated with a convex polyomino is word-representable if and only if G is 3-colorable. The method to prove this characterization theorem was essentially in showing that such a triangulation is 3-colorable if and only if it contains no wheel graph W_5 or W_7 as an induced subgraph (neither W_5 nor W_7 are word-representable).

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In this paper we extend the results of Akrobotu, Kitaev and Masárová [1] to the case of grid-covered cylinder graphs, which is a cyclic version of rectangular grid graphs; see Section 2.2 for definitions. It turns out that in this case, some of the graphs in question with chromatic number 4 are actually word-representable; for example, see the underlying graph in Fig. 3.7. Still, assuming that there are at least four sectors in a grid-covered cylinder graph, word-representable triangulations of such graphs are characterized by avoidance of W_5 and W_7 as induced subgraphs. On the other hand, we can also characterize word-representability of triangulations of grid-covered cylinder with three sectors as those avoiding the six graphs in Fig. 4.11 as induced subgraphs. Moreover, we show that our characterization results in the case of more than three sectors hold even when some of cells (faces) of grid-covered cylinder graphs are not triangulated.

The paper is organized as follows. In Section 2 we will provide all necessary definitions and known results to be used. In particular, we will introduce the notion of a triangulation of a grid-covered cylinder graph, the main concern of this paper. Also, we will introduce the notion of a semi-transitive orientation, the main tool to prove our results. Further, we classify word-representable triangulations of the graphs in question depending on the number of sectors they have. Namely, in Section 3 we will consider the case of grid-covered cylinder graphs with more than three sections, and in Section 4 we will consider the case of grid-covered cylinder graphs with three sections. Finally, in Section 5 we discuss a generalization of our results and state an open problem.

2. Definitions, notation, and known results

Suppose that w is a word and x and y are two distinct letters in w. We say that x and y *alternate* in w if the deletion of all other letters from the word w results in either $xyxy\cdots$ or $yxyx\cdots$.

A graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if $(x, y) \in E$ for each $x \neq y$. We say that w *represents* G, and such a word w is called a *word-representant* for G. For example, if the word $w = 134231241$ then the subword induced with letters 1 and 2 is 12121, hence the letters 1 and 2 alternate in w, and thus the respective vertices are connected in G. On the other hand, the letters 1 and 3 do not alternate in w, because removing all other letters we obtain 1331; thus, 1 and 3 are not connected in G. Fig. 2.1 shows the graph represented by w.

2.1. Semi-transitive orientations

A directed graph (digraph) is *semi-transitive* if it is acyclic (that is, it contains no directed cycles), and for any directed path $v_1 \to v_2 \to \cdots \to v_k$ with $v_i \in V$ for all i, $1 \leq i \leq k$, either

- there is no edge $v_1 \to v_k$, or
- the edge $v_1 \to v_k$ is present and there are edges $v_i \to v_j$ for all $1 \leq i < j \leq k$. That is, in this case, the (acyclic) subgraph induced by the vertices $v_1, \ldots, v_k$ is transitive.

We call such an orientation a *semi-transitive orientation*.

We can alternatively define semi-transitive orientations in terms of induced subgraphs. A *semi-cycle* is the directed acyclic graph obtained by reversing the direction of one arc of a directed cycle. An acyclic digraph is a *shortcut* if it is induced by the vertices of a semi-cycle and contains a pair of non-adjacent vertices. Thus, a digraph on the vertex set $\{v_1, \ldots, v_k\}$ is a shortcut if it contains a directed path $v_1 \to v_2 \to \cdots \to v_k$, the arc $v_1 \to v_k$, and it is missing an arc $v_i \to v_j$ for some $1 \leq i < j \leq k$; in particular, we must have $k \geq 4$, so that any shortcut is on at least four vertices. Slightly abusing the terminology, in this paper we refer to the arc $v_1 \to v_k$ in the last definition as a shortcut (a more appropriate name for this would be a *shortcut arc*). Fig. 2.2 gives examples of shortcuts, where the edges $1 \to 4$, $2 \to 5$ and $3 \to 6$ are missing, and hence $1 \to 5$, $1 \to 6$ and $2 \to 6$ are shortcuts.

Thus, an orientation of a graph is semi-transitive if it is acyclic and contains no shortcuts. Halldórsson, Kitaev and Pyatkin [8] proved the following theorem that characterizes word-representable graphs in terms of graph orientations.

Theorem 2.1 ([7]). *A graph is word-representable if and only if it admits a semi-transitive orientation.*

Thus, in this paper, to find out if a graph G in question is word-representable, we will be studying existence of a semi-transitive orientation of G.

An immediate corollary of Theorem 2.1 is the following result.

Theorem 2.2 ([7]). *Three-colorable graphs are word-representable.*

2.2. Grid-covered cylinder graphs

A *grid-covered cylinder*, *GCC* for brevity, is a 3-dimensional figure formed by drawing vertical lines and horizontal circles on the surface of a cylinder, each of which are parallel to the generating line and the upper face of the cylinder, respectively. A GCC can be thought of as the object obtained by gluing the left and right sides of a rectangular grid. See the left picture in Fig. 2.3 for a schematic way to draw a GCC. The vertical lines and horizontal circles are called the *grid lines* by us.