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Facial packing edge-coloring of plane graphs

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ABSTRACT

A *facial k-packing edge-coloring* of a plane graph G is a coloring of its edges with colors $1, \ldots, k$ such that every facial trail containing two edges with the same color i has length at least $i+2$. The smallest integer k such that G admits a facial k-packing edge-coloring is denoted by $p'_f(G)$. We prove that $p'_f(G) \leq 20$ for every 3-edge-connected plane graph G.

1. Introduction

Consider a simple graph G. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A *k-packing vertex-coloring* of G is a function $c : V(G) \to \{1, \ldots, k\}$ with the property that vertices in the ith class have pairwise distance greater than i, in other words, it is a partition of the set $V(G)$ into disjoint subsets $V_1, \ldots, V_k$ (some set can be empty) such that any two vertices from the set V_i are at distance at least $i+1$ in G. The smallest k for which G admits a k-packing vertex-coloring is called the *packing chromatic number*. A *packing edge-coloring* of G is defined as a packing vertex-coloring of its line graph $L(G)$.

The concept of packing chromatic number was introduced by Goddard et al. [7] under the name broadcast chromatic number. The term packing chromatic number was later proposed by Brešar et al. [1]. The packing vertex-coloring problem is known to be NP-complete on general graphs [7], moreover it remains NP-complete even for trees [4]. There have been written several papers on this concept, especially on the values for grids, lattices and distance graphs [1–3,5–7,9,10].

In this paper we consider the facial packing edge-coloring problem of plane graphs which can be considered as a relaxation of the packing edge-coloring problem. We focus on facial trails of plane graphs. A *facial k-packing edge-coloring* of a plane graph G is a coloring of its edges with colors $1, \ldots, k$ such that every facial trail containing two edges with the same color i has length at least $i+2$. The smallest integer k such that G admits a facial k-packing edge-coloring is denoted by $p'_f(G)$. The number $p'_f(G)$ is called the *facial packing index* of G. Observe that this coloring need not be proper in a usual way. We require only that *facially adjacent* edges (consecutive edges of a facial trail of some face) must receive different colors. On the other hand these types of colorings coincide in the class of paths and cycles.

For plane triangulations the facial packing edge-coloring problem is equivalent to the four color problem, see e.g. the book of Saaty and Kainen [12]. From the Four Color Theorem the following result follows, see [12, p. 103].

Theorem 1. *The edges of any plane triangulation T can be colored with 3 colors so that the edges bounding every face are colored distinctly, i.e.*

$$p'_f(T) = 3.$$

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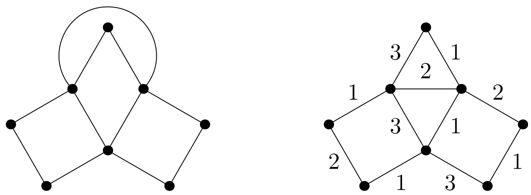

Fig. 1. Two embeddings of the same graph with different facial packing indices.

Shannon [14] proved that every multigraph G with maximum degree Δ has a proper edge-coloring with at most $\frac{3}{2}\Delta$ colors. This result can be reformulated for the family of plane graphs in the following way.

Theorem 2. *Let G be a 2-edge-connected plane graph with maximum face size Δ^*. Then the edges of G can be colored with at most $\frac{3}{2}\Delta^*$ colors in such a way that the edges bounding every face of G are colored distinctly.*

Corollary 1. *If G is a 2-edge-connected plane graph with maximum face size Δ^*, then*

$$p'_f(G) \le \frac{3}{2}\Delta^*.$$

Vizing [16,15] proved that simple planar graphs with maximum degree at least eight have the chromatic index (edge chromatic number) equal to their maximum degree. He conjectured the same if the maximum degree is either seven or six. The first part of this conjecture was proved by Sanders and Zhao [13]. Note that (also by Vizing) every graph with maximum degree Δ has the chromatic index equal to Δ or $\Delta + 1$. These results of Sanders and Zhao and of Vizing can be reformulated in the following way.

Theorem 3. *Let G be a 3-edge-connected plane graph with maximum face size $\Delta^* \ge 7$. Then the edges of G can be colored with Δ^* colors in such a way that the edges bounding every face of G are colored distinctly.*

Corollary 2. *If G is a 3-edge-connected plane graph with maximum face size $\Delta^* \ge 7$, then*

$$p'_f(G) \le \Delta^*.$$

In this paper, we show that there is a constant K such that $p'_f(G) \le K$ for every 3-edge-connected plane graph G.

2. Results

Note that the facial packing index depends on the embedding of the graph G. For example, the graph depicted in Fig. 1 with the embedding on the left has facial packing index at least four (the boundary of the outer face is a 7-cycle and therefore on its edges at most one edge is colored with 3, at most two edges have color 2 and at most three edges receive color 1); whereas with the embedding on the right, it has a facial packing edge-coloring with colors 1, 2, 3.

The *dual* G^* of a plane graph G can be obtained as follows: Corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*; two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G (an edge separates the faces incident with it).

We say that an edge-coloring of a plane graph is *suitable* if the following holds for every vertex: Let v be a vertex of degree k and let $e_1, \dots, e_k$ be the edges incident with v, listed in their clockwise order around v. If two edges e_i and e_j, $i < j \le k$, have the same color ℓ, then $j - i > \ell$ and $k - j + i > \ell$. This means that there are at least ℓ edges "between" any two edges of color ℓ.

Observation 1. *A 3-edge-connected plane graph G has a facial packing edge-coloring with colors $1, \dots, k$ if and only if its dual G^* has a suitable edge-coloring with colors $1, \dots, k$.*

This observation will play a major role in the proof of the main result. Instead of facial packing edge-coloring of G we shall investigate suitable edge-coloring of G^*.

Lemma 1. *Every star admits a suitable edge-coloring with colors* 1, 2, 3, 16.

Proof. Let S_n denote the star on n edges. If $n \le 3$, then we color the edges of S_n with n different colors. Now assume that $n \ge 4$. If n is a multiple of 4, then we color the edges of S_n with colors $1, 2, 1, 3, \dots, 1, 2, 1, 3$. When n is not a multiple of 4, then we distinguish three cases.

- If $n = 4k + 1$, then we repeat k times the pattern 1, 2, 1, 3 and then use color 16.
- If $n = 4k + 2$, then we repeat k times the pattern 1, 2, 1, 3 and then use colors 1, 16.
- If $n = 4k + 3$, then we repeat k times the pattern 1, 2, 1, 3 and then use colors 1, 2, 16.

Clearly, this coloring is a suitable edge-coloring. □