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Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On incidence coloring conjecture in Cartesian products of graphs

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ARTICLE INFO

Article history:
Received 9 December 2015
Received in revised form 18 April 2016
Accepted 23 April 2016
Available online xxxx

Keywords:
Incidence coloring
Cartesian product
Locally injective homomorphism

ABSTRACT

An *incidence* in a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident to v. Two incidences (v, e) and (u, f) are *adjacent* if at least one of the following holds: (a) $v = u$, (b) $e = f$, or (c) $vu \in \{e, f\}$. An *incidence coloring* of G is a coloring of its incidences assigning distinct colors to adjacent incidences. It was conjectured that at most $\Delta(G) + 2$ colors are needed for an incidence coloring of any graph G. The conjecture is false in general, but the bound holds for many classes of graphs. We introduce some sufficient properties of the two factor graphs of a Cartesian product graph G for which G admits an incidence coloring with at most $\Delta(G) + 2$ colors.

1. Introduction

An *incidence* in a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident to v. The set of all incidences of G is denoted $I(G)$. Two incidences (v, e) and (u, f) are *adjacent* if at least one of the following holds: (a) $v = u$, (b) $e = f$, or (c) $vu \in \{e, f\}$ (see Fig. 1). An *incidence coloring* of G is a coloring of its incidences such that adjacent incidences are assigned distinct colors. The least k such that G admits an incidence coloring with k colors is called the *incidence chromatic number of* G, denoted by $\chi_i(G)$. An incidence coloring of G is called *optimal* if it uses precisely $\chi_i(G)$ colors.

The incidence coloring of graphs was defined in 1993 by Brualdi and Massey [3] and attracted considerable attention as it is related to several other types of colorings. As already observed by the originators, it is directly connected to *strong edge-coloring*, i.e. a proper edge-coloring such that the edges at distance at most two receive distinct colors. Indeed, consider a bipartite graph H with the vertex set $V(H) = V(G) \cup E(G)$ and two vertices $v \in V(G)$ and $e \in E(G)$ adjacent in H if and only if v is incident to e in G; that is, H is the graph G with every edge subdivided. Then, a strong edge-coloring of H corresponds to an incidence coloring of G. This in turn means that the incidence chromatic number of a graph G is equal to the strong chromatic index of H.

A graph G is called a *$(\Delta + k)$-graph* if it admits an incidence coloring with $\Delta(G) + k$ colors for some positive integer k. A complete characterization of $(\Delta + 1)$-graphs is still an open problem. While it is a trivial observation that complete graphs and trees are such, it is harder to determine additional classes. This problem has already been addressed in several papers; it was shown that Halin graphs with maximum degree at least 5 [18], outerplanar graphs with maximum degree at least 7 [14], planar graphs with girth at least 14 [2], and square, honeycomb and hexagonal meshes [8] are $(\Delta + 1)$-graphs. In fact, every n-regular graph with a partition into $n + 1$ (perfect) dominating sets is such, as observed by Sun [17].

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http://dx.doi.org/10.1016/j.dam.2016.04.030

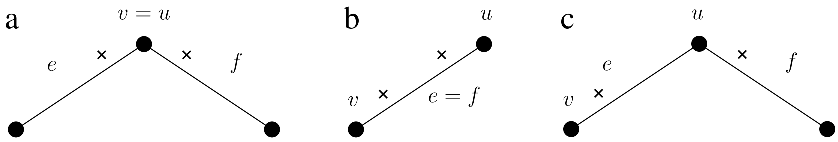

Fig. 1. Three types of adjacent incidences.

Theorem 1 (*Sun, 2012*). *If G is an n-regular graph, then $\chi_i(G) = n+1$ if and only if $V(G)$ is a disjoint union of $n+1$ dominating sets.*

Observation 1. *In any optimal incidence coloring c of a regular $(\Delta + 1)$-graph G, for every vertex v there is a color c_v such that for every edge uv, it holds $c(u, uv) = c_v$.*

Similarly intriguing as the lower bound is the upper bound. Brualdi and Massey [3] proved that $\chi_i(G) \leq 2\Delta(G)$ for every graph G. Aside to that, they proposed the following.

Conjecture 1 (*Brualdi and Massey, 1993*). *For every graph G,*

$$\chi_i(G) \leq \Delta(G) + 2.$$

Conjecture 1 has been disproved by Guiduli [6] who observed that incidence coloring is a special case of directed star arboricity, introduced by Algor and Alon [1]. Based on this observation, it was clear that Paley graphs are counterexamples to the conjecture. Nevertheless, Guiduli [6] decreased the upper bound for simple graphs.

Theorem 2 (*Guiduli, 1997*). *For every simple graph G,*

$$\chi_i(G) \leq \Delta(G) + 20 \log \Delta(G) + 84.$$

Although Conjecture 1 has been disproved in general, it has been confirmed for many graph classes, e.g. cubic graphs [10], partial 2-trees (and thus also outerplanar graphs) [7], and powers of cycles (with a finite number of exceptions) [11], to list just a few. We refer an interested reader to [15] for a thorough online survey on incidence coloring results.

Recently, Pai et al. [12] considered incidence coloring of hypercubes. Recall that the n-dimensional hypercube Q_n for an integer $n \geq 1$ is the graph with the vertex set $V(Q_n) = \{0, 1\}^n$ and edges between vertices that differ in exactly one coordinate. They proved that the conjecture holds for hypercubes of special dimensions in the following form.

Theorem 3 (*Pai et al., 2014*). *For every integers $p, q \geq 1$,*

(i) $\chi_i(Q_n) = n + 1$, *if* $n = 2^p - 1$;
(ii) $\chi_i(Q_n) = n + 2$, *if* $n = 2^p - 2$ *and* $p \geq 2$, *or* $n = 2^p + 2^q - 1$, *or* $n = 2^p + 2^q - 3$ *and* $p, q \geq 2$.

They also obtained some additional upper bounds for the dimensions of other forms and proposed the following conjecture.

Conjecture 2 (*Pai et al., 2014*). *For every $n \geq 1$ such that $n \neq 2^p - 1$ for every integer $p \geq 1$,*

$$\chi_i(Q_n) = n + 2.$$

Motivated by their research, we consider incidence coloring of Cartesian products of graphs; in particular we study sufficient conditions for the factors such that their Cartesian product is a $(\Delta + 2)$-graph. Conjecture 2 has recently been confirmed also by Shiau, Shiau and Wang [13], who also considered Cartesian products of graphs, but in a different setting, not limiting to $(\Delta + 2)$-graphs.

In Section 3, we show that if one of the factors is a $(\Delta + 1)$-graph and the maximum degree of the second is not too small, Conjecture 1 holds. As a corollary, Conjecture 2 is also answered in affirmative. In Section 4, we introduce two classes of graphs, 2-permutable and 2-adjustable, such that the Cartesian product of factors from each of them is a $(\Delta + 2)$-graph. In Section 5, we discuss further work and propose several problems and conjectures.

2. Preliminaries

In this section, we present additional terminology used in the paper. The *Cartesian product of graphs G and H*, denoted by $G \square H$, is the graph with the vertex set $V(G) \times V(H)$ and edges between vertices (u, v) and (u', v') whenever

- $uu' \in E(G)$ and $v = v'$ (a *G-edge*), or
- $u = u'$ and $vv' \in E(H)$ (an *H-edge*).

Please cite this article in press as: P. Gregor, et al., On incidence coloring conjecture in Cartesian products of graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.04.030