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Fractional zero forcing via three-color forcing games

Leslie Hogben [a,b,*], Kevin F. Palmowski [a], David E. Roberson [c], Michael Young [a]

[a] *Department of Mathematics, Iowa State University, Ames, IA 50011, USA*

[b] *American Institute of Mathematics, 600 E. Brokaw Rd., San Jose, CA 95112, USA*

[c] *Division of Mathematical Sciences, Nanyang Technological University, SPMS-MAS-03-01, 21 Nanyang Link, Singapore 637371, Singapore*

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ABSTRACT

An r-fold analogue of the positive semidefinite zero forcing process that is carried out on the r-blowup of a graph is introduced and used to define the fractional positive semidefinite forcing number. Properties of the graph blowup when colored with a fractional positive semidefinite forcing set are examined and used to define a three-color forcing game that directly computes the fractional positive semidefinite forcing number of a graph. We develop a fractional parameter based on the standard zero forcing process and it is shown that this parameter is exactly the skew zero forcing number with a three-color approach. This approach and an algorithm are used to characterize graphs whose skew zero forcing number equals zero.

1. Introduction

This paper studies fractional versions (in the spirit of [13]) of the standard and positive semidefinite zero forcing numbers and introduces three-color forcing games to compute these parameters. The three-color approach allows simpler proofs of some results and yields new results about existing parameters (see, e.g., Section 3.4).

The zero forcing process was introduced independently in [1] as a method of forcing zeros in a null vector of a symmetric matrix described by a graph, which yields an upper bound to the nullity of the matrix, and in [6] for control of quantum systems. There are potential applications to the spread of rumors or diseases (see, e.g., [5]); one of the original names of zero forcing was "graph infection". Despite the fact that when studied as a graph parameter there are no zeros involved, the name "zero forcing number" has become the standard term in the literature. The original zero forcing number has since spawned numerous variants (see, e.g., [3,4,11]). The speed with which the zero forcing process colors all vertices has also been studied (see, e.g., [9,14]).

1.1. Zero forcing games

In this section, we introduce several zero forcing processes, which can be described as coloring games [4], and necessary terminology. Abstractly, a *forcing game* is a type of coloring game that is played on a simple graph G. First, a non-white "target color," typically blue or dark blue, is designated. Each vertex of the graph is then colored white, the target color, or possibly some other color (in prior work, only white and the target color have been used). A *forcing rule* is chosen: this is a rule that

* Corresponding author at: Department of Mathematics, Iowa State University, Ames, IA 50011, USA.
E-mail addresses: hogben@iastate.edu, hogben@aimath.org (L. Hogben), kevin.palmowski@gmail.com (K.F. Palmowski), davideroberson@gmail.com (D.E. Roberson), myoung@iastate.edu (M. Young).

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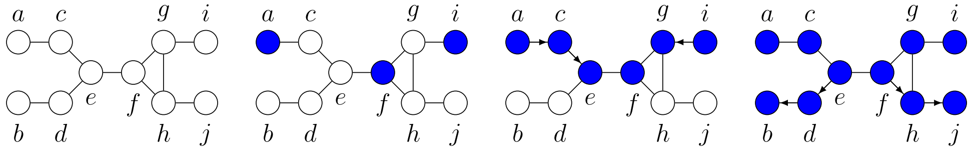
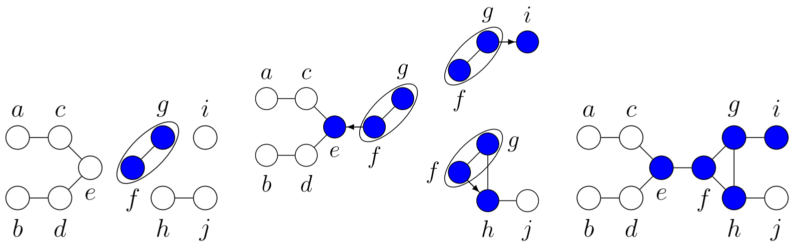


(a) Graph G. (b) Initial forcing set. (c) First three forces. (d) Final forces.

Fig. 1. Standard zero forcing game example.



(a) Connected components. (b) Forcing in each component. (c) Reassembled graph.

Fig. 2. Positive semidefinite zero forcing game example (first steps).

describes the conditions under which some vertex can cause another vertex to change to the target color. If vertex u causes a neighboring vertex w to change color, we say that u *forces* w and write $u \rightarrow w$. The forcing rule is repeatedly applied until no more forces can be performed, at which point the game ends; the coloring at the end is called the *final coloring*. An ordered list of the forces performed is referred to as a *chronological list of forces*. Note that there is usually some choice as to which forces are performed, as well as the order in which these forces occur. As such, a single forcing set may generate many different chronological lists of forces; however, the final coloring is unique for all of the games discussed herein. If the graph is totally colored with the target color at the end of the game, then we say that G has been *forced*. The goal of the game is to force G. If this is possible, then the initial set of non-white vertices is called a *forcing set*.

The *(standard) zero forcing game* uses only the colors blue (the target color) and white. The *(standard) zero forcing rule* is as follows:

> If w is the only white neighbor of a blue vertex u, then u can force w.

A *(standard) zero forcing set* is an initial set of blue vertices that can force G using this rule. The *(standard) zero forcing number* of G, denoted $Z(G)$, is the minimum cardinality of a zero forcing set for G. We present an illustrative example in Fig. 1.

From this point forward, we will omit the word "standard" when referring to the standard zero forcing game, its forcing rule, or zero forcing sets whenever there is no risk of ambiguity.

The *positive semidefinite zero forcing game* is a modification of the zero forcing game used to force zeros in a null vector of a positive semidefinite matrix described by a graph [3]. Like the zero forcing game, positive semidefinite zero forcing uses only the colors blue (target) and white. The *positive semidefinite zero forcing rule* is

> Let B be the set of blue vertices. Let $W_1, \dots, W_k$ be the sets of vertices of the k components of $G - B$ (it is possible that $k = 1$). If $u \in B$, $w \in W_i$, and w is the only white neighbor of u in $G[W_i \cup B]$, then change the color of w to blue.

If there is only one component, then we simply force via the standard forcing rule. If disconnection occurs when B is deleted, then after the force the graph is "reassembled" prior to applying the rule again. As one would expect, a *positive semidefinite zero forcing set* is an initial set of blue vertices that can force G using this rule, and the *positive semidefinite zero forcing number* of G, denoted $Z^+(G)$, is the minimum cardinality of a positive semidefinite zero forcing set for G. In Fig. 2 we illustrate the positive semidefinite zero forcing process on the graph from Fig. 1(a).

The *skew zero forcing game*, another variant on zero forcing that uses the colors white and blue (target), was first considered in [11] to force zeros in a null vector of a skew symmetric matrix described by a graph. The *skew zero forcing rule* is as follows:

> If w is the only white neighbor of any vertex u, then u can force w.

Skew zero forcing removes the standard requirement that the forcing vertex u be blue; as a result, skew zero forcing allows *white vertex forcing*, i.e., a white vertex is allowed to force its only white neighbor. A *skew zero forcing set* is an initial set of blue vertices that can force G using this rule, and the *skew zero forcing number* of G, denoted $Z^-(G)$, is the minimum cardinality of a skew zero forcing set for G. Fig. 3 demonstrates skew zero forcing; notice that the initial forcing set contains no blue vertices.