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Improving the lower bound on opaque sets for equilateral triangle☆

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ABSTRACT

An opaque set (or a barrier) for $U \subseteq \mathbb{R}^2$ is a set B of finite-length curves such that any line intersecting U also intersects B. In this paper, we consider the lower bound on the shortest barrier when U is the unit-size equilateral triangle. The best known lower bound is $3/2$, which comes from the classical fact that the length of the shortest barrier for any convex shape is at least the half of its perimeter. While such a general lower bound is slightly improved very recently, its applicability range does not cover the case of triangles. The main result of this paper is to find out this missing piece in part: We give the lower bound of $3/2 + 5 \cdot 10^{-13}$ for the unit-size equilateral triangle. The proof is based on two new ideas, angle-restricted barriers and a weighted sum of projection-cover conditions, which may be of independent interest.

1. Introduction

An opaque set (or a barrier) for $U \subseteq \mathbb{R}^2$ is a set $B \subseteq \mathbb{R}^2$ such that any line intersecting U also intersects B. A simple example is that given any geometric shape (e.g., square, triangle, and so on), its boundary forms a barrier. Note that we do not assume B is contained in U. Our main interest is to identify the barrier of a given shape U with the minimum total length. While we cannot define the total length of B in general because a connected piece $P \subseteq B$ may have a volume, P can be replaced by its boundary cycle without losing the property of barriers. Hence we assume that the volume of B is always zero without loss of generality. Then the length of B is well-defined. More precisely, we consider *rectifiable* barriers, consisting of countably many finite-length curves which are pairwise disjoint with each other except at the endpoints. Furthermore, a general and useful principle is that any rectifiable barrier with total length less than l implies the existence of a straight-line barrier composed of a countably finite number of line segments with length less than l [10]. Thus throughout this paper, we may focus only on straight barriers.

The problem of the shortest barrier is so classic, which is first posed by Mazurkiewicz in 1916 [11]. Surprisingly, even for simple polygons such as squares or triangles, the length of the shortest barrier is still not identified. Only the lower bounds, which are probably not tight, are currently known: A general lower bound has been shown by Jones in 1964 [9], which exhibits the fact that the shortest barrier for any convex polygon must be longer than the half of its perimeter. That is, the shortest barrier for the unit-size square must be at least two, and for the unit-size equilateral triangle it must be at least $3/2$. After that, the problem was revived in several times [2,4,8], and there are a number of papers considering its algorithmic aspects [1,3,6,7,12]. This paper focuses more on the mathematical aspect: We argue explicit lower bounds for a specific shape U.

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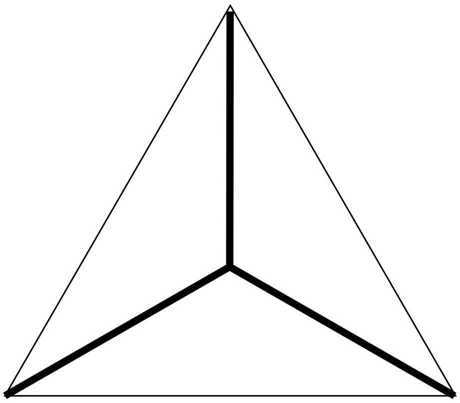

Fig. 1. The barrier O (bold lines) conjectured to be optimal.

For explicit lower bounds beyond Jones' one, very recently, two papers propose improved lower bounds for squares [5,10]. The result by [5] is conditional, which assumes that any segment in the barrier is not so far from the boundary of the square. The paper by Kawamura et al. [10] gives an unconditional lower bound of 2.0002 for the unit-size square. Furthermore, they show that any convex shape other than triangles with perimeter $2p$, there exists a constant ϵ (depending on the shape) such that $p + \epsilon$ becomes a lower bound for the barrier. Unfortunately, this result does not apply to triangles. The best known lower bound for the unit-size equilateral triangle is still $3/2$. In this paper, we improve this lower bound by a small constant. More precisely, we obtain the lower bound of length $3/2 + 5 \cdot 10^{-13}$. While it is still far from the currently best barrier O with length $\sqrt{3}$ (Fig. 1), which is conjectured to be optimal, this result is the first non-trivial improvement of Jones' bound for equilateral triangles.

The following part of the paper is organized as follows: In Section 2, we state several notations and our proof ideas, which include the proof of Jones' bound. Our proof is divided into two subcases. Sections 3 and 4 correspond to those cases, and they are integrated in Section 5. Finally, we conclude this paper in Section 6.

2. Preliminaries and proof outline

Throughout this paper, we use the term "equilateral triangle" as the meaning of "unit-size equilateral triangle". We assume that any barrier B considered in this paper is a straight barrier, and thus regard B as a (possibly infinite) set of segments. Note that this assumption is not essential: By Kawamura et al. [10], it is shown that getting a lower bound for straight barriers implies getting the same bound for general (unconditional) barriers. For $X \subseteq \mathbb{R}^2$, we define $X(\alpha) \in \mathbb{R}$ as the image of X projected onto the line with angle α passing the origin. Precisely, $X(\alpha) = \{x \cos\alpha + y \sin\alpha | (x, y) \in X\}$. For any set X of segments, we denote by $|X|$ the sum of the length of all the segments in X, and denote by $|X(\alpha)|$ the sum of the length of the segments constituting the image $X(\alpha)$. We have the following necessary condition:

$$\forall\alpha \in [0, \pi] : |U(\alpha)| \leq |B(\alpha)| \leq \sum_{l \in B} |l(\alpha)|.$$

That is, for any angle α, the projection of U must be covered by the projection of B. Otherwise, B cannot be a barrier because there exists a line orthogonal to the plane with angle α intersecting U but not intersecting B. We call this inequality the *projection-cover condition* for α.

The bound by Jones [9] is obtained by summing up the projection-cover conditions for all $\alpha \in [0, \pi]$.

$$p = \int_0^\pi |U(\alpha)| \mathrm{d}\alpha \leq \int_0^\pi |B(\alpha)| \mathrm{d}\alpha \leq \sum_{l \in B} |l| \cdot \int_0^\pi |\cos\alpha| \mathrm{d}\alpha = 2|B|, \tag{1}$$

where p is the perimeter of U. In the case that U is the equilateral triangle, $p = 3$. Note that the first equality is obtained by Cauchy's surface area formula.

Our lower bound proof is based on two new ideas. The first one is to consider *angle-restricted* barriers: Letting $A \subseteq [0, \pi]$, we say that B is A-restricted if any segment $l \in B$ has an angle in A. Given an A-restricted barrier and an angle $\phi \in A$, we denote the set of segments in B with angle ϕ by B_ϕ.

The next idea is an extension of Jones' bound to obtain better bounds for angle-restricted barriers. The key observation behind the extension is an interpretation of Jones' bound in the context of linear programming. Let U be a convex polygon, $L(\alpha)$ be the set of lines with angle $\alpha + \pi/2$ intersecting U, and $L = \cup_\alpha L(\alpha)$. Now we define any segment by their two endpoints, that is, we regard a segment as an element in $(\mathbb{R}^2)^2$. The length of a segment s is denoted by $|s|$. For segment s, we also define a 0–1 variable x_s. Letting X_l be the set of segments intersecting a line $l \in L$, the constraint that the line l