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# Fractional spanning tree packing, forest covering and eigenvalues

Yanmei Hong [a], Xiaofeng Gu [b,*], Hong-Jian Lai [c], Qinghai Liu [d]

[a] College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian 350108, China
[b] Department of Mathematics, University of West Georgia, Carrollton, GA 30118, USA
[c] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA
[d] Center for Discrete Mathematics, Fuzhou University, Fuzhou, Fujian 350002, China

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ABSTRACT

We investigate the relationship between the eigenvalues of a graph $G$ and fractional spanning tree packing and coverings of $G$. Let $\omega(G)$ denote the number of components of a graph $G$. The strength $\eta(G)$ and the fractional arboricity $\gamma(G)$ are defined by

$$\eta(G) = \min \frac{|X|}{\omega(G-X)-\omega(G)}, \quad \text{and} \quad \gamma(G) = \max \frac{|E(H)|}{|V(H)|-1},$$

where the optima are taken over all edge subsets $X$ whenever the denominator is non-zero. The well known spanning tree packing theorem by Nash-Williams and Tutte indicates that a graph $G$ has $k$ edge-disjoint spanning tree if and only if $\eta(G) \geq k$; and Nash-Williams proved that a graph $G$ can be covered by at most $k$ forests if and only if $\gamma(G) \leq k$. Let $\lambda_i(G)$ ($\mu_i(G)$, $q_i(G)$, respectively) denote the $i$th largest adjacency (Laplacian, signless Laplacian, respectively) eigenvalue of $G$. In this paper, we prove the following.

(1) Let $G$ be a graph with $\delta \geq 2s/t$. Then $\eta(G) \geq s/t$ if $\mu_{n-1}(G) > \frac{2s-1}{t(\delta+1)}$, or if $\lambda_2(G) < \delta - \frac{2s-1}{t(\delta+1)}$, or if $q_2(G) < 2\delta - \frac{2s-1}{t(\delta+1)}$.

(2) Suppose that $G$ is a graph with nonincreasing degree sequence $d_1, d_2, \ldots, d_n$ and $n \geq \lfloor \frac{2s}{t} \rfloor + 1$. Let $\beta = \frac{2s}{t} - \frac{1}{\lfloor \frac{2s}{t} \rfloor + 1} \sum_{i=1}^{\lfloor \frac{2s}{t} \rfloor + 1} d_i$. Then $\gamma(G) \leq s/t$, if $\beta \geq 1$, or if $0 < \beta < 1$, $n > \lfloor 2s/t \rfloor + 1 + \frac{2s-2}{t\beta}$ and

$$\mu_{n-1}(G) > \frac{n(2s/t - 2/t - \beta(\lfloor 2s/t \rfloor + 1))}{(\lfloor 2s/t \rfloor + 1)(n - \lfloor 2s/t \rfloor - 1)}.$$

Our result proves a stronger version of a conjecture by Cioabă and Wong on the relationship between eigenvalues and spanning tree packing, and sharpens former results in this area.


* Corresponding author.
*E-mail address:* xgu@westga.edu (X. Gu).

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## 1. Introduction

In this paper, we consider finite undirected simple graphs. Throughout the paper, $k, s, t$ denote positive integers and $G$ denotes a simple graph. We follow the notations of Bondy and Murty [1], unless otherwise defined. However, we use $\omega(G)$ to denote the number of components of $G$, which differs from [1].

Let $G$ be an undirected simple graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The **adjacency matrix** of $G$ is an $n$ by $n$ matrix $A(G)$ with entry $a_{ij} = 1$ if there is an edge between $v_i$ and $v_j$ and $a_{ij} = 0$ otherwise, for $1 \le i, j \le n$. We use $\lambda_i(G)$ to denote the $i$th largest eigenvalue of $G$; when the graph $G$ is understood from the context, we often use $\lambda_i$ for $\lambda_i(G)$. With these notations, we always have $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. Let $D(G) = (d_{ij})$ be the **degree matrix** of $G$, that is, the $n$ by $n$ diagonal matrix with $d_{ii}$ being the degree of vertex $v_i$ in $G$ for $1 \le i \le n$. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the **Laplacian matrix** and the **signless Laplacian matrix** of $G$, respectively. We use $\mu_i(G)$ and $q_i(G)$ to denote the $i$th largest eigenvalue of $L(G)$ and $Q(G)$, respectively. It is not difficult to see that $\mu_n(G) = 0$. The second smallest eigenvalue of $L(G)$, $\mu_{n-1}(G)$, is known as the **algebraic connectivity** of $G$.

For a connected graph $G$, the **spanning tree packing number**, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees in $G$. The **arboricity** $a(G)$ is the minimum number of edge-disjoint forests whose union equals $E(G)$. Fundamental theorems characterizing graphs $G$ with $\tau(G) \ge k$ and with $a(G) \le k$ have been obtained by Nash-Williams and Tutte, and by Nash-Williams, respectively.

**Theorem 1.1.** *Let $G$ be a connected graph with $E(G) \neq \emptyset$. Each of the following holds.*

(i) *(Nash-Williams [12] and Tutte [15]). $\tau(G) \ge k$ if and only if for any $X \subseteq E(G)$, $|X| \ge k(\omega(G - X) - 1)$.*

(ii) *(Nash-Williams [13]). $a(G) \le k$ if and only if for any subgraph $H$ of $G$, $|E(H)| \le k(|V(H)| - 1)$.*

Following the terminology in [3,14], we define the **strength** $\eta(G)$ and the **fractional arboricity** $\gamma(G)$ of a graph $G$ respectively by

$$\eta(G) = \min \frac{|X|}{\omega(G - X) - \omega(G)}, \quad \text{and} \quad \gamma(G) = \max \frac{|E(H)|}{|V(H)| - 1},$$

where the optima are taken over all edge subsets $X$ whenever the denominator is non-zero. Theorem 1.1 indicates that for a connected graph $G$, $\tau(G) \ge k$ if and only if $\eta(G) \ge k$, and, $a(G) \le k$ if and only if $\gamma(G) \le k$. Since $\eta(G)$ and $\gamma(G)$ are possibly fractional, we have $\tau(G) = \lfloor \eta(G) \rfloor$ and $a(G) = \lceil \gamma(G) \rceil$. Thus, $\eta(G)$ is also referred to as the **fractional spanning tree packing number** of $G$.

Cioabă and Wong [4] investigated the relationship between the second largest adjacency eigenvalue and $\tau(G)$ for a regular graph $G$, and made Conjecture 1.1(i). Utilizing Theorem 1.1, Cioabă and Wong proved Conjecture 1.1(i) for $k \in \{2, 3\}$.

Conjecture 1.1(i) was then extended to Conjecture 1.1(ii) for any simple graph $G$ (not necessarily regular). See [5–7,9,11] for the conjecture and some partial results. Recently, Conjecture 1.1 was settled in [10].

**Conjecture 1.1.** (i) *([4]) Let $k$ and $d$ be two integers with $d \ge 2k \ge 4$. If $G$ is a $d$-regular graph with $\lambda_2(G) < d - \frac{2k-1}{d+1}$, then $\tau(G) \ge k$.*

(ii) *([5,7,9,11]) Let $k$ be an integer with $k \ge 2$ and $G$ be a graph with minimum degree $\delta \ge 2k$. If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \ge k$.*

Motivated by the above conjecture and the corresponding results, we investigate the relationship between $\eta(G)$ and eigenvalues of $G$. We also consider the relationship between the fractional arboricity $\gamma(G)$ and algebraic connectivity $\mu_{n-1}(G)$. Theorems 1.2 and 1.3 are the main results.

**Theorem 1.2.** *Let $G$ be a graph with $\delta \ge 2s/t$.*

(i) *If $\mu_{n-1}(G) > \frac{2s-1}{t(\delta+1)}$, then $\eta(G) \ge s/t$.*

(ii) *If $\lambda_2(G) < \delta - \frac{2s-1}{t(\delta+1)}$, then $\eta(G) \ge s/t$.*

(iii) *If $q_2(G) < 2\delta - \frac{2s-1}{t(\delta+1)}$, then $\eta(G) \ge s/t$.*

**Remark 1.** Theorem 1.2 indicates that, for a graph $G$ with $\delta \ge 2k$, if $\mu_{n-1}(G) > \frac{2k-1}{\delta+1}$, or $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, or $q_2(G) < 2\delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \ge k$. This was proved in [10] and settled Conjecture 1.1.

**Theorem 1.3.** *Suppose that $G$ is a graph with nonincreasing degree sequence $d_1, d_2, \ldots, d_n$ and $n \ge \lfloor \frac{2s}{t} \rfloor + 1$. Let $\beta = \frac{2s}{t} - \frac{1}{\lfloor \frac{2s}{t} \rfloor + 1} \sum_{i=1}^{\lfloor \frac{2s}{t} \rfloor + 1} d_i$.*