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Note

Proof of a conjecture on the zero forcing number of a graph[☆]

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ABSTRACT

Amos et al. (2015) introduced the notion of the k-forcing number of graph for a positive integer k as the generalization of the zero forcing number of a graph. The k-forcing number of a simple graph G, denoted by $F_k(G)$, is the minimum number of vertices that need to be initially colored so that all vertices eventually become colored during the discrete dynamical process defined by the following rule. Starting from an initial set of colored vertices and stopping when all vertices are colored: if a colored vertex has at most k non-colored neighbors, then each of its non-colored neighbors become colored. Particularly, with a close connection to the maximum nullity of a graph, $F_1(G)$ is widely studied under the name of the zero forcing number, denoted by $Z(G)$. Among other things, Amos et al. proved that for a connected graph G of order n with $\Delta = \Delta(G) \geq 2$, $Z(G) \leq \frac{(\Delta-2)n+2}{\Delta-1}$, and this inequality is sharp. Moreover, they conjectured that $Z(G) = \frac{(\Delta-2)n+2}{\Delta-1}$ if and only if $G = C_n$, $G = K_{\Delta+1}$ or $G = K_{\Delta,\Delta}$. In this note, we show the above conjecture is true.

1. Introduction

We consider undirected finite simple connected graphs only. For notation and terminology not defined here, we refer to [6]. For a graph $G = (V(G), E(G))$, $|V(G)|$ and $|E(G)|$ are its order and size, respectively. For a vertex $v \in V(G)$, the neighborhood $N(v)$ of v is defined as the set of vertices adjacent to v. The degree $d_G(v)$ of v is the number of edges incident with v in G. The minimum and maximum degrees of a vertex in a graph G are denoted $\delta(G)$ and $\Delta(G)$, respectively. Let $S \subseteq V(G)$. Denote the set of the edges between S and $\overline{S}$ by $E(S, \overline{S})$, and let $e(S, \overline{S}) = |E(S, \overline{S})|$. The subgraph induced by S, denoted by $G[S]$, is the graph with vertex set S, in which two vertices x and y are adjacent if and only if they are adjacent in G. As usual, for a positive integer $n \geq 1$, K_n and $K_{n,n}$ denote respectively the complete graph of order n and the complete bipartite graph with n vertices in its each part; C_m denote the cycle of order m for an integer $m \geq 3$.

Next, we follow the definition by Amos et al. [2]. Let k be a positive integer and G a graph. A set $S \subseteq V(G)$ is a *k-forcing set* if, when its vertices are initially colored – while the remaining vertices are initially non-colored – and the graph is subjected to the following color change rule, all of vertices in G will eventually become colored. A colored vertex with at most k non-colored neighbors will cause each the non-colored neighbor to become colored. The *k-forcing number* of G, denoted by $F_k(G)$, is the cardinality of the smallest k-forcing set. If a vertex u causes a vertex v to change colors during the k-forcing process, we say that u *k-forces* v (in particular, u forces v when $k = 1$).

This concept generalizes a widely studied notion of the zero forcing number $Z(G)$ of a graph G. Indeed, $F_1(G) = Z(G)$. Barioli et al. [3] and Burgarth et al. [7] introduced independently the concepts of zero forcing set and zero forcing number of a graph. In [3], it is introduced to bound the maximum nullity of a graph. Namely, for a graph G whose vertices are labeled

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from 1 to n, $M(G)$ denotes the maximum nullity over all symmetric real valued matrices where, for $i \neq j$, the ijth entry in nonzero if and only if ij is an edge in G. Then, $M(G) \leq Z(G)$ for any graph G. For the more results on the relation between the relation of the maximum nullity and the zero forcing number of a graph, we refer to [4,5,9–15]. In [7], the zero forcing set of a graph has been used in order to study the controllability of quantum systems. Aazami [1] proved the NP-hardness of computing the zero forcing number of a graph, using a reduction from the Directed Hamiltonian Cycle problem.

Amos et al. [2] generalized the concept of zero forcing number of a graph to the k-forcing number of a graph for an integer $k \geq 1$ and proved that for a connected graph G of order n with $\Delta = \Delta(G) \geq 2$, $Z(G) \leq \frac{(\Delta-2)n+2}{\Delta-1}$, and this inequality is sharp. Moreover, they posed the following conjecture.

Conjecture 1.1 (*Amos et al. [2]*). *Let G be a connected graph with $\Delta \geq 2$. Then*

$$Z(G) = \frac{(\Delta-2)n+2}{\Delta-1},$$

if and only if $G = C_n$, $G = K_{\Delta+1}$ or $G = K_{\Delta,\Delta}$.

In this note, we confirm the validity of the above conjecture.

2. Some results on $Z(G)$

A *k-dominating set* of a graph G is a set D of vertices such that every vertex not in D is adjacent to at least k vertices in D.

Lemma 2.1 (*Lemma 4.1 in [2]*). *Let k be a positive integer and $G = (V, E)$ be a k-connected graph with $n > k$. If S is a smallest k-forcing set such that the subgraph induced by $V \setminus S$ is connected, then $V \setminus S$ is a connected k-dominating set of G.*

Theorem 2.2 (*Theorem 4.4 in [2]*). *Let k be positive integer and let $G = (V, E)$ be a k-connected graph with $n > k$ vertices and $\Delta \geq 2$. Then*

$$F_k(G) \leq \frac{(\Delta-2)n+2}{\Delta+k-2},$$

and this inequality is sharp.

For the special case of $k = 1$, the above bound was improved by Caro and Pepper as follows.

Theorem 2.3 (*Corollary 3.1 in [8]*). *Let G be a connected graph of order n with maximum degree Δ and minimum degree δ. Then*

$$Z(G) = F_1(G) \leq \frac{(\Delta-2)n-(\Delta-\delta)+2}{\Delta-1}.$$

Lemma 2.4. *Let T be a tree with exactly k leaves. If S is a set of $k-1$ leaves of T, then S is a zero forcing set of T.*

Proof. The proof is by induction on k. If $k = 2$, T is path, and the result clearly holds. Now assume that $k \geq 3$. Take a vertex $u \in S$. Let P be a maximal path of T containing u such that every vertex v on P has degree at most two in T. Let $T' = T - V(P)$. Note that T' has exactly $k-1$ leaves. By the induction hypothesis, $S' = S \setminus \{u\}$ is a zero forcing set of T'. So, S is a zero forcing set of T. □

3. Main result

Theorem 3.1. *Let G be a connected graph with $\Delta \geq 2$. Then*

$$Z(G) = \frac{(\Delta-2)n+2}{\Delta-1},$$

if and only if $G = C_n$, $G = K_{\Delta+1}$ or $G = K_{\Delta,\Delta}$.

Proof. It is clear that $Z(C_n) = 2$ for any $n \geq 3$, $Z(K_{\Delta+1}) = \Delta$, $Z(K_{\Delta,\Delta}) = 2\Delta - 2$. Hence, the sufficiency of theorem holds trivially.

To show the necessity, we assume that G is a connected graph of order n with $\Delta \geq 2$ and $Z(G) = \frac{(\Delta-2)n+2}{\Delta-1}$. By Theorem 2.3, G is a Δ-regular graph. If $\Delta = 2$, then $G = C_n$. In what follows, we assume that $\Delta \geq 3$.

Let S be a smallest zero forcing set of G such that $G[\overline{S}]$ is connected, where $\overline{S} = V \setminus S$. Thus,

$$|S| \geq Z(G) = \frac{(\Delta-2)n+2}{\Delta-1}. \tag{1}$$