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On Subset Families That Form a Continuous Lattice

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Abstract

It is well known that continuous lattices and algebraic lattices can be respectively represented by the family of all fixed points of the projection operator and the closure operator preserving sups of directed sets on the power set of a set X. Similar to the algebraic $\bigcap$-structure as the concrete representation of algebraic lattices, can we have a the concrete representation of continuous lattices by families of sets? We give a positive answer to this in this paper. Also, as a special type of continuous lattice, a concrete representation of completely distributive complete lattices by families of sets is obtained.

Keywords: continuous lattices, completely distributive, $\mathbb{C}$-sets, $\mathbb{A}$-sets, $\mathbb{C}$-$\bigcup$-semiring.

1 Introduction

In Lattice Theory, the representations of special type of lattices are always important research topics. Many approaches, such as in terms of families of subsets of a set, topological spaces, formal concepts and information systems [4,5,9,12,13,15], have been used for representing special type of lattices. The most intuitive one is by means of families of subsets of a set.

In [2], Buchi showed that a lattice is complete if and only if it is isomorphic to a topped $\bigcap$-structure (intersection structure). Reney proposed the concept of the ring of sets and revealed that the completely distributive algebraic lattice (completely supercontinuous lattice) are one-to-one correspondent to the ring of sets in [13]. After then, Deng [5] suggested the conclusion that a complete lattice is completely distributive if and only if it is isomorphic to a complete $\bigcup$-semiring with condition

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($*$). Also in [3], the author proved that a lattice is algebraic if and only if it is isomorphic to an algebraic $\bigcap$-structure. As a generalization of algebraic $\bigcap$-structure, the notion of F-algebraic $\bigcap$-structure by Guo and Li in [9]. It has been showed that a dcpo is an algebraic domain if and only if it is isomorphic to an F-algebraic $\bigcap$-structure.

As a more general result, it is proved in [8] that a lattice L is continuous iff it is isomorphic to the family of all fixed points of a projection operator preserving directed sups. Thus the family of all fixed points of a projection operator preserving directed sups can be viewed as a representation of continuous lattices, but it is difficult to justify whether a family of sets satisfy the conditions. In this paper, our purpose is to obtain a concrete representation of continuous lattices, by means of families of subsets of a set, similar to the algebraic $\bigcap$-structure as the concrete representation of the algebraic lattice.

The rest of this paper is structured as follows. In Section 2, we recall some basic concepts in lattice theory which will be frequently used in this paper. In Section 3, we obtain the representations of continuous lattices (algebraic lattices, completely distributive lattices, respectively). In Section 4, we consider a special family of sets: the Scott topology on a poset. Then we can get some equivalent conditions for a poset to be continuous.

2 Preliminaries

A partially ordered set (poset) is a nonempty set P equipped with a reflexive, transitive and antisymmetric relation $\leq$. Let P be a poset and $X \subseteq P$, we use the symbol $\sup X = \bigvee X$ to denote the least upper bound of X, and $\inf X = \bigwedge X$ for the greatest lower bound of X. We denote $\uparrow X = \{x \in P : \exists y \in X, s.t. y \leq x\}$, $\downarrow X = \{x \in P : \exists y \in X, s.t. x \leq y\}$, $X^l = \{x \in P : \forall y \in X, x \leqslant y\}$ and $X^u = \{x \in P : \forall y \in X, y \leq x\}$.

A subset D of P is said to be directed provided it is nonempty and every finite subset of D has an upper bound in D. Dually, we call a nonempty subset F of P filtered if every finite subset of F has a lower bound in F.

Definition 2.1 *[8] Let P be a poset. We say that x is way below y, in symbols $x \ll y$, iff for all directed subsets $D \subseteq P$ for which $\sup D$ exists, the relation $y \leq \sup D$ implies the existence of an element $d \in D$ with $x \leq d$. An element satisfying $x \ll x$ is said to be compact, the set of all compact elements is denoted by $K(P)$.*

For each $x \in P$, we simply write $\twoheaddownarrow x = \{y \in P \mid y \ll x\}$, $\twoheaduparrow x = \{y \in P \mid x \ll y\}$. It is easy to see that $\twoheaddownarrow x$ is a lower subset of P and $\twoheaduparrow x$ is an upper subset of P for each $x \in P$.

Definition 2.2 *[8] 1) A poset P is called continuous if for all $x \in P$, the set $\twoheaddownarrow x = \{y \in P \mid y \ll x\}$ is directed and $x = \sup \twoheaddownarrow x$.*
2) A dcpo which is continuous as a poset is called a domain.
3) A domain which is a complete lattice is called a continuous lattice.