Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science 325 (2016) 3–27

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Binding Operators for Nominal Sets

Arthur Azevedo de Amorim[1]

Department of Computer and Information Science
University of Pennsylvania
Philadelphia, PA, United States

Abstract

The theory of nominal sets is a rich mathematical framework for studying syntax and variable binding. Within it, we can describe several binding disciplines and derive convenient reasoning principles that respect α-equivalence. In this article, we introduce the notion of *binding operator*, a novel construction on nominal sets that unifies and generalizes many forms of binding proposed in the literature. We present general results about these operators, including sufficient conditions for validly using them in inductive definitions of nominal sets.

Keywords: Nominal Sets, Binding, Alpha Equivalence

1 Introduction

Bound variables have puzzled computer scientists and logicians for decades. Although fairly simple to handle in informal pencil-and-paper calculations, they can be surprisingly complex to manage in algorithms and mechanized proofs, where the mostly uninteresting formal details of variable binding cannot be overlooked. Research on the subject has led to various promising approaches for tackling this complexity [6,12,14], among which we can mention the theory of *nominal sets* [5].

Nominal sets constitute a rich mathematical universe where objects contain *variables* that can be *renamed*, allowing various notions of α-equivalence to be defined. In the λ-calculus for example, we stipulate that the term $\lambda x.\, t$ is equivalent to any other obtained by renaming x to a variable y that does not appear free in t, which corresponds to the operation of *name abstraction* on nominal sets [5], used for modeling objects with a single bound variable. The nominal literature has shown how many other forms of binding can be obtained through similar constructions, such

[1] The author thanks Andrew Pitts, Ranald Clouston, and Matthew Weaver for comments on early drafts of this article. This work is supported by NSF award 1513854, *Micro-Policies: A Framework for Tag-Based Security Monitors*.

http://dx.doi.org/10.1016/j.entcs.2016.09.029
1571-0661/

as *generalized* name abstractions [4,3], or the binding declarations of Nominal Isabelle [17]. Besides serving as a good theoretical foundation for variable binding, nominal techniques have influenced the design of many tools for manipulating syntax, such as the FreshML programming language [11,15] and the Nominal package for Isabelle/HOL [16,17].

Although some of the notions above are more general than others, none of them proposes to offer a clear, unified picture of what binding means for nominal sets. In this article, we attempt to look at the problem from a more foundational perspective, by introducing *binding operators*: a novel construction on nominal sets that unifies and generalizes many forms of variable binding proposed in the literature. After briefly recalling basic notions of nominal set theory (Section 2), we introduce binding operators in Section 3, showing how to use them for defining a variety of nominal sets representing binders in Section 4. Section 5 gives an alternative characterization of these nominal sets defined by binding operators, used in Section 6 to encompass variable scope within our framework. In Section 7, we discuss category-theoretic properties of binding operators, which provide sufficient conditions for defining nominal sets inductively. We conclude and review related work in Section 8.

2 Preliminaries: Nominal Sets

We begin by recalling basic concepts and results of the theory of nominal sets; for a detailed account on the subject, we refer the reader to the introductory article by Gabbay and Pitts [5] or to Pitts' book [10].

We fix some countably infinite set $\mathbb{A}$. We refer to elements of $\mathbb{A}$ as *atoms*, and use the variable a to denote them. A *permutation* of $\mathbb{A}$ is a bijective function $\pi : \mathbb{A} \to \mathbb{A}$ such that $\pi(a) = a$ for all but finitely many $a \in \mathbb{A}$. Permutations form a group under composition, noted $\mathrm{perm}(\mathbb{A})$; in particular, $\pi \circ \pi' \in \mathrm{perm}(\mathbb{A})$ and $\pi^{-1} \in \mathrm{perm}(\mathbb{A})$ for every $\pi, \pi' \in \mathrm{perm}(\mathbb{A})$.

A *renaming operation* on a set X is a *group action* of $\mathrm{perm}(\mathbb{A})$ on X. Spelled out explicitly, this means a mapping that to each pair $(\pi, x) \in \mathrm{perm}(\mathbb{A}) \times X$ associates an element $\pi \cdot x \in X$, so that

$$1 \cdot x = x \qquad\qquad (\pi_1 \circ \pi_2) \cdot x = \pi_1 \cdot \pi_2 \cdot x,$$

where $1 \in \mathrm{perm}(\mathbb{A})$ denotes the identity function. We treat renaming as right associative, reading $\pi_1 \cdot \pi_2 \cdot x$ as $\pi_1 \cdot (\pi_2 \cdot x)$. The above properties imply in particular $\pi^{-1} \cdot \pi \cdot x = \pi \cdot \pi^{-1} \cdot x = x$ for arbitrary π and x.

We say that a set of atoms A *supports* an element $x \in X$ if the atoms in A completely determine the effect of renaming on x. Formally, if π is a permutation such that $\pi(a) = a$ for every a in A, then $\pi \cdot x = x$. Or, equivalently, if π_1 and π_2 are permutations such that $\pi_1(a) = \pi_2(a)$ for every a in A, then $\pi_1 \cdot x = \pi_2 \cdot x$. If A is finite, we can show [5] that x has a *minimal* finite supporting set $\mathrm{supp}(x)$, by which we mean that $\mathrm{supp}(x)$ is a subset of every finite set A' supporting x. We say that X is a *nominal set* if all of its elements have finite support.