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Giry and the Machine

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Abstract

We present a general method - the Machine - to analyse and characterise in finitary terms natural transformations between well-known functors in the category **Pol** of Polish spaces. The method relies on a detailed analysis of the structure of **Pol** and a small set of categorical conditions on the domain and codomain functors. We apply the Machine to transformations from the Giry and positive measures functors to combinations of the Vietoris, multiset, Giry and positive measures functors. The multiset functor is shown to be defined in **Pol** and its properties established. We also show that for some combinations of these functors, there cannot exist more than one natural transformation between the functors, in particular the Giry monad has no natural transformations to itself apart from the identity. Finally we show how the Dirichlet and Poisson processes can be constructed with the Machine.

Keywords: probability, topology, category theory, monads

1 Introduction

Classical tools of probability theory are not geared towards compositionality, and especially not compositional approximation (Kozen, [13]). This has not prevented authors from developing powerful techniques (Chaput et al. [5], Kozen et al. [14]) based on structural approaches to probability theory (Giry, [9]). Here, we adopt a slightly different standpoint: we propose to tackle this tooling problem *globally*, by combining structural insights of **Pol** together with some classical tools of probability theory and topology put in functorial form. The outcome is the Machine, an axiomatic reconstruction in category-theoretic terms of developments carried out in [7]. Thus, we get a simpler and more conceptual proof of our previous results. We also obtain a much more comprehensive picture and prove that natural transformations between Giry-like functors are entirely characterised by their components on

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finite spaces. For instance, the monadic data of the Giry functor are easily obtained from the finite case (which is completely elementary) and applying the Machine. But the construction is not limited to probability functors: we deal similarly with the multiset and the Vietoris (the topological powerdomain of compact subsets) functors. This allows one to consider transformations mixing probabilistic and ordinary non-determinism, in a way which is reminiscent of (Keimel et al., [12]). Another byproduct of our Machine is that we reconstruct from finitary data classical objects of probability theory and statistics, namely the Poisson and Dirichlet processes. It is worth noting that Poisson, Dirichlet (and many other similar constructions obtained by recombining the basic ingredients differently) are obtained as natural and continuous maps: naturality expresses the stability of the "behaviour" in a change of granularity, and as such is a fundamental property of consistency, but continuity (which to our knowledge is proved here for the first time) expresses a no less important property, namely the robustness of the behaviour in changes in "parameters". This has potential implications in Bayesian learning.

The structure of the paper is as follows. In Sec. 3, we show that **Pol** is stratified into the subcategories $\mathbf{Pol}_f$, $\mathbf{Pol}_{cz}$ $\mathbf{Pol}_z$ of finite, compact zero-dimensional and zero-dimensional Polish spaces respectively and show how these subcategories are related. In Sec. 4, the Machine is introduced: we identify a small set of categorical conditions on functors F, G that guarantee that any natural transformation from F to G in $\mathbf{Pol}_f$ can be extended step-by-step through the subcategories to a natural transformation on **Pol**. In Sec. 5, we illustrate the workings of the Machine on natural transformations connecting the Giry and positive measure functors to combinations of the Vietoris, multiset, Giry and positive measure functors. As far as we know, the multiset functor is defined in **Pol** for the first time and its properties are established. As a first application of the Machine, we develop in Sec. 6 general criteria under which there can exist at most one natural transformation from a functor F to the Giry functor. In particular, we show that there exists at most one natural transformation between the Vietoris, multiset, positive measure and Giry functor to the Giry functor. Lastly, we show in Sec. 7 how transformations of the type $\mathsf{M}^+ \Rightarrow \mathsf{G}H$ where M^+ is the finite measure functor and H is either the multiset or the finite measure functor can be built in $\mathbf{Pol}_f$ from a single generating morphism $\mathsf{M}^+(1) \to \mathsf{G}H(1)$ and give criteria for this transformation to be natural. In particular, we show that the Dirichlet and Poisson distributions satisfy these criteria and use the Machine to build Dirichlet and Poisson processes.

2 Notations

Most of our developments take place in the category **Pol** of Polish spaces and continuous maps. **Pol** is a full subcategory of the category **Top** of topological spaces and continuous maps. **Pol** has all countable limits and all countable coproducts (Bourbaki [4], IX). The functor mapping any space to the measurable space having the same underlying set and the Borel σ-algebra and interpreting continuous maps as measurable ones will be denoted by $\mathcal{B} : \mathbf{Pol} \to \mathbf{Meas}$, where **Meas** is the category