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On the Representation of Semigroups and Other Congruences in the Lambda Calculus

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Abstract

We show that every semigroup with an RE word problem can be pointwise represented in the lambda calculus. In addition, we show that the free monoid generated by an arbitrary RE subset of combinators can be represented as the monoid of all terms which fix a finite set of points.

Keywords: lambda calculus, semigroups, representation

1 Introduction

Combinators being both functions and arguments can act on one another by application and composition. More generally, if $\$'$ and \$" are sets of combinators closed under beta conversion, the A action of $\$'$ on \$" is the set $\{AMN|M : \$'$ and N:\$"} closed under beta conversion. First we recall the definitions of some familiar combinators:

$$\begin{aligned} B &:= \lambda abc.a(bc) & S &:= \lambda abc.ac(bc) \\ B' &:= \lambda abc.b(ac) & O &:= (\lambda x.xx)(\lambda x.xx) \\ C^* &:= \lambda ab.ba & 0 &:= \lambda yz.z \\ K &:= \lambda ab.a & s &:= \lambda xyz.y(xyz) \\ I &:= \lambda a.a & Y &:= (\lambda xz.z(xxz))(\lambda xyz.z(xxz)) \end{aligned}$$

Example 1.1 $A := K$: this is the trivial action.

Example 1.2 $A := I$: this is the applicative action.

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Example 1.3 $A := B$: this is the semigroup action.

Example 1.4 $A := S$: the pointwise applicative action.

Of course, this definition extends to multiple arguments by Currying. We write

$$M = N \text{ mod beta}$$

if M beta converts to N.

It is trivial that general A can be reduced to I, and that multiple arguments can be reduced to a single argument by pairing. In addition, applicative action can be reduced to the semigroup action since $K(xy) = Bx(Ky)$ mod beta. However, there is another reduction which is lambda I.

Let

$$D := Y(\lambda fxyz.\ fx(zy))$$

where Y is Turing's fixed point combinator as above.

Lemma 1.5 *For any* U, V *if* $B(C^*U)D = B(C^*V)D$ mod beta *then*

$$U = V \text{ mod beta.}$$

Proof. Straightforward. □

Now given $\$'$ and $\$''$, since

$$\begin{aligned} C^*(AM) &= B(B(C^*M)(C^*A))B \text{ mod beta} \\ C^*(AMN) &= B(B(B(C^*M)(C^*A))B)(B(C^*N)D) \text{ mod beta, and} \\ &= B(C^*M)(B(C^*A)(BB(B(C^*N)D))) \text{ mod beta} \end{aligned}$$

the A action of $\$'$ on $\$''$ is equivalent to the semigroup action of $\{C^*M \mid M : \$'\}$ on $\{(B(C^*A)(BB(B(C^*N)D))) \mid N : \$''\}$. We next consider an example of the action of I in representing semi-groups.

Definition 1.6 Let $\$'$ be an RE set of combinators closed under beta conversion. An equivalence relation $\sim$ on $\$'$ is said to be pointwise representable on $\$''$ if for every $M, N : \$'$ we have

$$MP : \$'' \text{ for all } P : \$''$$

$$M \sim N \text{ iff for all } P : \$'' \quad MP = NP \text{ mod beta}$$

Example 1.7 (Kleene):

$\$'$ = any RE set of definitions of total recursive functions

$\$''$ = the Church numerals and $\sim =$ extensional equality

Non-example (Plotkin):