Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science 324 (2016) 165–179

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

A Unified Procedure for Provability and Counter-Model Generation in Minimal Implicational Logic

Jefferson de Barros Santos[1,2]

Departamento de Informática
PUC-Rio
Brazil

Bruno Lopes Vieira[4]

Instituto de Computação
UFF
Brazil

Edward Hermann Haeusler[3]

Departamento de Informática
PUC-Rio
Brazil

Abstract

This paper presents results on the definition of a sequent calculus for Minimal Implicational Propositional Logic ($\mathbf{M}^{\rightarrow}$) aimed to be used for provability and counter-model generation in this logic. The system tracks the attempts to construct a proof in such a way that, if the original formula is a $\mathbf{M}^{\rightarrow}$ tautology, the tree structure produced by the proving process is a proof, otherwise, it is used to construct a counter-model using Kripke semantics.

Keywords: theorem proving

1 Introduction

Proof search (validity) in Minimal Implicational Propositional Logic ($\mathbf{M}^{\rightarrow}$) is a PSPACE-Complete problem as stated by Statman in [14] who also shows that

[1] The authors thank to CNPq and CAPES for supporting this research

[2] Email:jsantos@inf.puc-rio.br

[3] Email:hermann@inf.puc-rio.br

[4] Email:bruno@ic.uff.br

http://dx.doi.org/10.1016/j.entcs.2016.09.014
1571-0661/

$\mathbf{M}^{\rightarrow}$ polynomially simulates Intuitionistic Propositional Logic (**IPL**). This simulation can be used to polynomially simulate Classical Logic too, although Classical Logic seems to be in a smaller complexity class [5]. This result points out that $\mathbf{M}^{\rightarrow}$ is as hard to implement as the most popular propositional logics. Haeusler shows in [7] that $\mathbf{M}^{\rightarrow}$ can polynomially simulate not only Classical and Intuitionistic Propositional Logic but also Full Minimal Propositional Logic and any other decidable propositional logic with a Natural Deduction system with the sub-formula property. Because of those features, $\mathbf{M}^{\rightarrow}$ can be used as a base tool to study complexity of many other logics.

Our goal here is to present a *sequent calculus* for $\mathbf{M}^{\rightarrow}$ which allows the definition of a unified procedure for provability and counter-model generation in this logic. The calculus is based on a set of rules and in a general strategy for application of the rules in such a way that we can avoid the usage of loop checkers and the necessity of working with different systems for provability and refutation. To the best of our knowledge, this is the first proof calculus for $\mathbf{M}^{\rightarrow}$ where validity and counter-model generation are done in a single procedure.

Counter-model generation (using Kripke semantics) is achieved as a consequence of the completeness of the system. We are also developing an interactive theorem prover for $\mathbf{M}^{\rightarrow}$ based on the in here proposed calculus. Its source code can be found at `https://github.com/jeffsantos/sequent-prover`.

2 Minimal Implicational Logic

2.1 Semantics

The Minimal Implicational Logic ($\mathbf{M}^{\rightarrow}$) is the fragment of Minimal Logic containing only the logical constant $\rightarrow$. Its semantics is the intuitionistic semantics restricted to $\rightarrow$ only. Thus, given a propositional language $\mathcal{L}$, a $\mathbf{M}^{\rightarrow}$ model is a structure $\langle U, \preceq, \mathcal{V}\rangle$, where U is a non-empty set (worlds), $\preceq$ is a partial order relation on U and $\mathcal{V}$ is a function from U into the power set of $\mathcal{L}$, such that if $i, j \in U$ and $i \preceq j$ then $\mathcal{V}(i) \subseteq \mathcal{V}(j)$. Given a model, the satisfaction relationship $\models$ between worlds in models and formulae is defined as in Intuitionistic Logic, namely:

- $\langle U, \preceq, \mathcal{V}\rangle \models_i p$, $p \in \mathcal{L}$, iff, $p \in \mathcal{V}(i)$
- $\langle U, \preceq, \mathcal{V}\rangle \models_i \alpha_1 \rightarrow \alpha_2$, iff, for every $j \in U$, such that $i \preceq j$, if $\langle U, \preceq, \mathcal{V}\rangle \models_j \alpha_1$ then $\langle U, \preceq, \mathcal{V}\rangle \models_j \alpha_2$.

As usual a formula α is valid in a model $\mathcal{M}$, namely $\mathcal{M} \models \alpha$, if and only if, it is satisfiable in every world i of the model, namely $\forall i \in U, \mathcal{M} \models_i \alpha$. A formula is a $\mathbf{M}^{\rightarrow}$ tautology, if and only if, it is valid in every model.

2.2 Syntax

It is known that Prawitz Natural deduction system for Minimal Logic with only the $\rightarrow$-rules ($\rightarrow$-Elim and $\rightarrow$-Intro) is sound and complete for the $\mathbf{M}^{\rightarrow}$ regarding Kripke

[5] We remember that we do not know whether $NP = PSPACE$ or not.