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Hitting forbidden subgraphs in graphs of bounded treewidth ☆

Marek Cygan [a], Dániel Marx [b], Marcin Pilipczuk [c], Michał Pilipczuk [a]

[a] Institute of Informatics, University of Warsaw, Poland
[b] Institute for Computer Science and Control, Hungarian Academy of Sciences (MTA SZTAKI), Hungary
[c] Department of Computer Science, University of Warwick, United Kingdom

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ABSTRACT

We study the complexity of a generic hitting problem H-Subgraph Hitting, where given a fixed pattern graph H and an input graph G, the task is to find a set $X \subseteq V(G)$ of minimum size that hits all subgraphs of G isomorphic to H. In the colorful variant of the problem, each vertex of G is precolored with some color from $V(H)$ and we require to hit only H-subgraphs with matching colors. Standard techniques shows that for every fixed H, the problem is fixed-parameter tractable parameterized by the treewidth of G; however, it is not clear how exactly the running time should depend on treewidth. For the colorful variant, we demonstrate matching upper and lower bounds showing that the dependence of the running time on treewidth of G is tightly governed by $\mu(H)$, the maximum size of a minimal vertex separator in H. That is, we show for every fixed H that, on a graph of treewidth t, the colorful problem can be solved in time $2^{\mathcal{O}(t^{\mu(H)})} \cdot |V(G)|$, but cannot be solved in time $2^{o(t^{\mu(H)})} \cdot |V(G)|^{O(1)}$, assuming the Exponential Time Hypothesis (ETH). Furthermore, we give some preliminary results showing that, in the absence of colors, the parameterized complexity landscape of H-Subgraph Hitting is much richer.

1. Introduction

The "optimality programme" is a thriving trend within parameterized complexity, which focuses on pursuing tight bounds on the time complexity of parameterized problems. Instead of just determining whether the problem is fixed-parameter tractable, that is, whether the problem with a certain parameter k can be solved in time $f(k) \cdot n^{\mathcal{O}(1)}$ for some computable function $f(k)$, the goal is to determine the best possible dependence $f(k)$ on the parameter k. For several problems, matching upper and lower bounds have been obtained on the function $f(k)$. The lower bounds are under the complexity assumption Exponential Time Hypothesis (ETH), which roughly states than n-variable 3SAT cannot be solved in time $2^{o(n)}$; see, e.g., the survey of Lokshtanov et al. [1].

One area where this line of research was particularly successful is the study of fixed-parameter algorithms parameterized by the treewidth of the input graph and understanding how the running time has to depend on the treewidth. Classic results on model checking monadic second-order logic on graphs of bounded treewidth, such as Courcelle's Theorem, provide a unified and generic way of proving fixed-parameter tractability of most of the tractable cases of this parameterization [2,3]. While these results show that certain problems are solvable in time $f(t) \cdot n$ on graphs of treewidth t for some function f, the

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E-mail addresses: cygan@mimuw.edu.pl (M. Cygan), dmarx@cs.bme.hu (D. Marx), malcin@mimuw.edu.pl (Marcin Pilipczuk), michal.pilipczuk@mimuw.edu.pl (Michał Pilipczuk).

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exact function $f(t)$ resulting from this approach is usually hard to determine and far from optimal. To get reasonable upper bounds on $f(t)$, one typically resorts to constructing a dynamic programming algorithm, which often is straightforward, but tedious.

The question whether the straightforward dynamic programming algorithms for bounded treewidth graphs are optimal received particular attention in 2011. On the hardness side, Lokshtanov, Marx and Saurabh proved that many natural algorithms are probably optimal [4,5]. In particular, they showed that there are problems for which the $2^{\mathcal{O}(t\log t)}n$ time algorithms are best possible, assuming ETH. On the algorithmic side, Cygan et al. [6] presented a new technique, called *Cut&Count*, that improved the running time of the previously known (natural) algorithms for many connectivity problems. For example, previously only $2^{\mathcal{O}(t\log t)} \cdot n^{\mathcal{O}(1)}$ algorithms were known for Hamiltonian Cycle and Feedback Vertex Set, which was improved to $2^{\mathcal{O}(t)} \cdot n^{\mathcal{O}(1)}$ by Cut&Count. These results indicated that not only proving tight bounds for algorithms on tree decompositions is within our reach, but such a research may lead to surprising algorithmic developments. Further work includes derandomization of Cut&Count in [7,8], an attempt to provide a meta-theorem to describe problems solvable in single-exponential time [9], and a new algorithm for Planarization [10].

We continue here this line of research by investigating a family of subgraph-hitting problems parameterized by treewidth and find surprisingly tight bounds for a number of problems. An interesting conceptual message of our results is that, for every integer $c \geq 1$, there are fairly natural problems where the best possible dependence on treewidth is of the form $2^{\mathcal{O}(t^c)}$.

Studied problems and motivation In our paper we focus on the following generic H-Subgraph Hitting problem: for a pattern graph H and an input graph G, what is the minimum size of a set $X \subseteq V(G)$ that hits all subgraphs of G that are isomorphic to H? (Henceforth we call them *H-subgraphs* for brevity.) This problem generalizes a few other problems studied in the literature, for example Vertex Cover (for $H = P_2$) [4], or finding the largest induced subgraph of maximum degree at most Δ (for $H = K_{1,\Delta+1}$) [11]. We also study the following *colorful* variant Colorful H-Subgraph Hitting, where the input graph G is additionally equipped with a coloring $\sigma : V(G) \to V(H)$, and we are only interested in hitting H-subgraphs where every vertex matches its color.

A direct source of motivation for our study is the work of Pilipczuk [9], which attempted to describe graph problems admitting fixed-parameter algorithms with running time of the form $2^{\mathcal{O}(t)} \cdot |V(G)|^{\mathcal{O}(1)}$, where t is the treewidth of G. The proposed description is a logical formalism where one can quantify existence of some vertex/edge sets, whose properties can be verified "locally" by requesting satisfaction of a formula of modal logic in every vertex. In particular, Pilipczuk argued that the language for expressing local properties needs to be somehow modal, as it cannot be able to discover cycles in a constant-radius neighborhood of a vertex. This claim was supported by a lower bound: unless ETH fails, for any constant $\ell \geq 5$, the problem of finding the minimum size of a set that hits all the cycles C_ℓ in a graph of treewidth t cannot be solved in time $2^{o(t^2)} \cdot |V(G)|^{\mathcal{O}(1)}$. Motivated by this result, we think that it is natural to investigate the complexity of hitting subgraphs for more general patterns H, instead of just cycles.

We may see the colorful variant as an intermediate step towards full understanding of the complexity of H-Subgraph Hitting, but it is also an interesting problem on its own. It often turns out that the colorful variants of problems are easier to investigate, while their study reveals useful insights; a remarkable example is the kernelization lower bound for Set Cover and related problems [12]. In our case, if we allow colors, a major combinatorial difficulty vanishes: when the algorithm keeps track of different parts of the pattern H that appear in the graph G, and combines a few parts into a larger one, the coloring σ ensures that the parts are vertex-disjoint. Hence, the colorful variant is easier to study, whereas at the same time it reveals interesting insight into the standard variant.

Our results and techniques In the case of Colorful H-Subgraph Hitting, we obtain tight bounds for the complexity of the treewidth parameterization. First, note that, in the presence of colors, one can actually solve Colorful H-Subgraph Hitting for each connected component of H independently; hence, we may focus only on connected patterns H. Second, we observe that there are two special cases. If H is a path then Colorful H-Subgraph Hitting reduces to a maximum flow/minimum cut problem, and hence is polynomial-time solvable. If H is a clique, then any H-subgraph of G needs to be contained in a single bag of any tree decomposition, and there is a simple $2^{\mathcal{O}(t)}|V(G)|$-time algorithm, where t is the treewidth of G. Finally, for the remaining cases we show that the dependence on treewidth is tightly connected to the value of $\mu(H)$, the maximum size of a minimal vertex separator in H (a separator S is minimal if there are two vertices x, y such that S is an xy-separator, but no proper subset of S is). We prove the following matching upper and lower bounds.

Theorem 1. *A Colorful H-Subgraph Hitting instance (G, σ) can be solved in time $2^{\mathcal{O}(t^{\mu(H)})}|V(G)|$ in the case when H is connected and is not a clique, where t is the treewidth of G.*

Theorem 2. *Let H be a graph that contains a connected component that is neither a path nor a clique. Then, unless ETH fails, there does not exist an algorithm that, given a Colorful H-Subgraph Hitting instance (G, σ) and a tree decomposition of G of width t, resolves (G, σ) in time $2^{o(t^{\mu(H)})}|V(G)|^{\mathcal{O}(1)}$.*

In every theorem of this paper, we treat H as a fixed graph of constant size, and hence the factors hidden in the $\mathcal{O}$-notation may depend on the size of H.