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Deciding game invariance

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ABSTRACT

In a previous paper, Duchêne and Rigo introduced the notion of invariance for take-away games on heaps. Roughly speaking, these are games whose rulesets do not depend on the position. Given a sequence S of positive tuples of integers, the question of whether there exists an invariant game having S as set of $\mathcal{P}$-positions is relevant. In particular, it was recently proved by Larsson et al. that if S is a pair of complementary Beatty sequences, then the answer to this question is always positive. In this paper, we show that for a fairly large set of sequences (expressed by infinite words), the answer to this question is decidable.

1. Introduction

Let $n \geq 1$ be an integer. In this paper, we consider take-away impartial games played over n piles of tokens. Two players alternatively remove a positive number of tokens from one or several piles following a prescribed ruleset. The rules are the same for both players. We assume normal convention, i.e., the player making the last move wins. Since we always remove a positive number of tokens, the game is acyclic and there is always a winner.

A *position* of such a game is an n-tuple of non-negative integers which corresponds to the number of tokens available in each pile. A *move* is also an n-tuple of non-negative integers corresponding to the number of tokens that are removed from each pile. Let $\mathbf{p} = (p_1, \ldots, p_n)$ be a position and $\mathbf{m} = (m_1, \ldots, m_n)$ be a non-zero move. The move $\mathbf{m}$ can be applied to the position $\mathbf{p}$ provided that $\mathbf{m} \leq \mathbf{p}$, i.e., for all i, $m_i \leq p_i$. The position resulting of the application of $\mathbf{m}$ is the n-tuple $\mathbf{p} - \mathbf{m}$.

Definition 1. A *game*, played over n piles, is given by a function $G : \mathbb{N}^n \to 2^{\mathbb{N}^n}$ that maps every position $\mathbf{p}$ to a set of moves that can be chosen from $\mathbf{p}$ by the player. Otherwise stated, the ruleset is provided by the map G. For a position $\mathbf{p}$, the set of *options* of $\mathbf{p}$ is the set $\{\mathbf{p} - \mathbf{m} \mid \mathbf{m} \in G(\mathbf{p})\}$ of positions where the player can move directly. A *strategy* consists in choosing a particular option for every position.

An interval of integers is denoted by $[\![k, \ell]\!]$. For an example of take-away game, the game of Nim over 2 piles is described by the map

$$G_{\mathrm{NIM}} : \mathbb{N}^2 \to 2^{\mathbb{N}^2}, (x, y) \mapsto \{(i, 0) \mid i \in [\![1, x]\!]\} \cup \{(0, j) \mid j \in [\![1, y]\!]\}.$$

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For Wythoff's game, the description is given by

$$G_{\text{WYTHOFF}} : \mathbb{N}^2 \to 2^{\mathbb{N}^2}, (x, y) \mapsto G_{\text{NIM}}(x, y) \cup \{(k, k) \mid k \in [\![1, \min\{x, y\}]\!]\}.$$

With such a formal presentation, we recall the notion of invariant game introduced in [12]. Note that we shall later on distinguish two notions of invariance: invariant games and admissible subsets.

Definition 2. A game $G : \mathbb{N}^n \to 2^{\mathbb{N}^n}$ is *invariant* if there exists a set $I \subseteq \mathbb{N}^n$ such that, for all positions $\mathbf{p}$, we have

$$G(\mathbf{p}) = I \cap \{\mathbf{m} \in \mathbb{N}^n \mid \mathbf{m} \leq \mathbf{p}\}.$$

Otherwise stated, we may apply exactly the same moves to every position, with the only restriction that there are enough tokens left. Since a game is defined by its moves, formally by the map G, one also speaks of *invariant moves*.

A motivation to introduce the notion of invariance is the relative simplicity of the corresponding rulesets. Roughly speaking, one has "just" to remember the set I.

The game of Nim defined above is invariant. Simply consider the set

$$I_{\text{NIM}} = \{(i, 0) \mid i \geq 1\} \cup \{(0, j) \mid j \geq 1\}.$$

Similarly, Wythoff's game is invariant with the set

$$I_{\text{WYTHOFF}} = I_{\text{NIM}} \cup \{(k, k) \mid k \geq 1\}.$$

For an example of non-invariant game, consider the following map,

$$G_{\text{EVEN}} : \mathbb{N}^2 \to 2^{\mathbb{N}^2}, (x, y) \mapsto \begin{cases} \{(i, 0) \mid i \in [\![1, x]\!]\}, & \text{if } x + y \text{ is even;} \\ \{(i, i) \mid i \in [\![1, \min\{x, y\}]\!]\}, & \text{otherwise.} \end{cases}$$

Here, the moves that can be applied from a position (x, y) depend on the position itself.

Recently, Fraenkel and Larsson introduced a generalization of this notion of invariance [18].

Definition 3. Let $t \geq 1$ be an integer. A game $G : \mathbb{N}^n \to 2^{\mathbb{N}^n}$ is *t-invariant* if the set of positions can be partitioned into t subsets $S_1, \ldots, S_t$ and there exist t sets $I_1, \ldots, I_t \subseteq \mathbb{N}^n$ such that, for all positions $\mathbf{p}$,

$$\text{if } \mathbf{p} \in S_j, \text{ then } G(\mathbf{p}) = I_j \cap \{\mathbf{m} \in \mathbb{N}^n \mid \mathbf{m} \leq \mathbf{p}\}.$$

In particular, an invariant game is 1-invariant.

Example 4. The game G_{EVEN} is clearly 2-invariant. One considers the partition of $\mathbb{N}^2$ into $S_1 = \{(x, y) \mid x + y \text{ is even}\}$ and $S_2 = \{(x, y) \mid x + y \text{ is odd}\}$.

Note that there exist some games which are not t-invariant for any t.

Example 5. The game

$$G_{\text{MARK}} : \mathbb{N} \to 2^{\mathbb{N}}, x \mapsto \{1, \lceil x/2 \rceil\}$$

defined in [15] is not t-invariant for any t.

It is classical to associate a set of $\mathcal{P}$-positions with a game.

Definition 6. A position $\mathbf{p} \in \mathbb{N}^n$ is a *$\mathcal{P}$-position* if there exists a strategy for the second player (i.e., the player who will play on the next round) to win the game, whatever the move of the first player is. We let $\mathcal{P}(G)$ denote the set of $\mathcal{P}$-positions of the game G. Conversely, $\mathbf{p}$ is an *$\mathcal{N}$-position* if there exists a winning strategy for the first player (i.e., the one who is making the current move).

The characterization of the set of $\mathcal{P}$-positions of an impartial acyclic game is well-known.

Proposition 7. *The sets of $\mathcal{P}$- and $\mathcal{N}$-positions of an impartial acyclic game are uniquely determined by the following two properties:*

- *Every move from a $\mathcal{P}$-position leads to an $\mathcal{N}$-position (stability property of the set of $\mathcal{P}$-positions).*
- *From every $\mathcal{N}$-position, there exists a move leading to a $\mathcal{P}$-position (absorbing property of the set of $\mathcal{P}$-positions).*