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Weighted automata on infinite words in the context of Attacker–Defender games

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ABSTRACT

The paper is devoted to several infinite-state Attacker–Defender games with reachability objectives. We prove the undecidability of checking for the existence of a winning strategy in several low-dimensional mathematical games including vector reachability games, word games and braid games. To prove these results, we consider a model of weighted automata operating on infinite words and prove that the universality problem is undecidable for this new class of weighted automata. We show that the universality problem is undecidable by using a non-standard encoding of the infinite Post correspondence problem.

1. Introduction

In the last decade there has been a steady, growing interest in the area of infinite-state games and computational complexity of checking whether a winning strategy exists [1–7]. Such games provide a powerful mathematical framework for a large number of computational problems. In particular, they appear in the verification, refinement and compatibility checking of reactive systems [8], analysis of programs with recursion [4], combinatorial topology and have deep connections with automata theory and logic [6,7,9].

In many cases of high-dimensional games, the problem of checking for the existence of a winning strategy can be computationally hard and even undecidable. Answering the same question for low-dimensional systems can also be a challenging problem. This is either due to a lack of tools for analysis of complex dynamics or due to a lack of "space" to encode directly the universal computations to show that the problem is undecidable.

In this paper we present three variants of low-dimensional Attacker–Defender games (i.e., word games, matrix games and braid games) for which it is undecidable to determine whether one of the players has a winning strategy. In addition, the proof incorporates a new language theoretical result (Theorem 2) about weighted automata on infinite words that can be efficiently used in the context of other reachability games.

An Attacker–Defender game is played in rounds, where in each round a move of Defender (Player 1) is followed by a move of Attacker (Player 2) starting from some initial position. The aim of Attacker is to reach a target position while Defender tries to keep Attacker from reaching the target position. Then, we say that Attacker has a winning strategy if she can eventually reach the target position regardless of Defender's moves. We show that in a number of restricted cases of

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such games, it is not possible to decide whether a winning strategy exists for a given set of moves, an initial position and a target position.

In particular, we introduce matrix games on vectors, where we show that if both players are stateless and the moves correspond to very restricted linear transformations from $SL(4,\mathbb{Z})$, the existence of a winning strategy is undecidable. There exists a simple reduction from known undecidable reachability games (robot games [10]) that leads to undecidability for a game with linear transformations in dimension six. To prove the undecidability in four-dimensional games, we first show undecidability of *word games* where players are given words over a group alphabet and in an alternating, way concatenate their words with a goal for Attacker to reach the empty word. The games on words, over semigroup alphabets, are commonly used to prove results in language theory [11–13]. We, on the other hand, define word games over group alphabets.

Later we show that it is possible to stretch the application of the proposed techniques to other models and frameworks. For example, we consider games on braids, which were recently studied in [14,15]. Braids are classical topological objects that attracted a lot of attention due to their connections to topological knots and links, as well as their applications to polymer chemistry, molecular biology, cryptography, quantum computations and robotics [16–20]. In this paper we consider games on braids with only three or five strands, where the braid is modified by a composition of braids from a finite set with the target for Attacker to reach the trivial braid. We show that it is undecidable to check for the existence of a winning strategy for three strands from a given nontrivial braid and for five strands starting from the trivial braid. The reachability with a single player (i.e., with nondeterministic composition from a single set) was shown to be decidable for braids with three strands in [21].

The undecidability results of this paper are proved by using a new language-theoretic result showing that the universality problem for weighted automata $\mathcal{A}$ on infinite words is undecidable. The acceptance of an infinite word w intuitively means that there exists a finite prefix p of w such that for the word p there is a path in $\mathcal{A}$ that has zero weight. From an instance of the infinite Post correspondence problem we construct the automaton $\mathcal{A}$ that accepts all infinite words if and only if the instance does not have a solution. As the infinite Post correspondence is undecidable [22], so is the universality problem for weighted automata on infinite words.

The considered model of automaton is closely related to *integer weighted finite automata* as defined in [23] and [24], where finite automata are accepting finite words and have additive integer weights on the transitions. In [23], it was shown that the universality problem is undecidable for integer weighted finite automata on finite words by reduction from the Post correspondence problem. In the context of a game scenario it is important to define acceptance of infinite words (which represent infinite plays in games) by considering finite prefixes reaching a target value. On the other hand, non-acceptance means that there exists an infinite computational path where none of the finite prefixes reach the target value. Then the universality for weighted automata over infinite words is the property ensuring that all infinite words are accepted (i.e., eventually reach a target in a computation path). Please note that while the universality for weighted automata on finite words implies universality for weighted automata on infinite words, the statement does not hold the other way around. Therefore the universality problem for weighted automata on infinite words is not equivalent to the universality problem for weighted automata on finite words. Our undecidability proof for weighted automata on infinite words follows the initial idea from [23] for mapping computations on words into a weighted (one-counter) automata model, which is extended with new constructions and formal proofs.

The paper is organized as follows. The next section contains basic notations and preliminaries used in the rest of the paper. In the third section we prove our main result that the universality problem is undecidable for weighted automata on infinite words. This section is the most involved and provides a new non-standard encoding of the infinite Post correspondence problem into the universality problem. Finally, in Section 4, we apply the main result to several games on mathematical objects and show that it is undecidable to check whether one of the players has a winning strategy. After each game we provide an example to illustrate the main principles of the game. We also show that the Attacker–Defender games are quite sensitive to the process of determinization. We discuss and illustrate this effect in details in the case of matrix games. Moreover, following our analysis, we formulate a problem of eventual reachability that is relevant in the context of concurrent games.

2. Notation and definitions

Words An *infinite word* w over a finite alphabet A is an infinite sequence of letters, $w = a_0a_1a_2a_3\cdots$ where $a_i \in A$ is a letter for each $i = 0, 1, 2, \ldots$. We denote the set of all infinite words over A by A^ω. The monoid of all finite words over A is denoted by A^*. A word $u \in A^*$ is a *prefix* of $v \in A^*$, denoted by $u \le v$, if $v = uw$ for some $w \in A^*$. If u and w are both nonempty, then the prefix u is called *proper*, denoted by $u < v$. A *prefix* of an infinite word $w \in A^\omega$ is a finite word $p \in A^*$ such that $w = pw'$ where $w' \in A^\omega$. This is also denoted by $p \le w$. The length of a finite word u is denoted by $|u|$. By $w[i]$ we denote the ith letter of a word w, i.e., $w = w[1]w[2]\cdots$. The *reversed word* of $w = w[1]\cdots w[n]$ is denoted by $w^R = w[n]\cdots w[1]$, i.e., the order of the letters is reversed. Later in the paper the finite words will be denoted by u, v, infinite words by w and single letters by a, b, c, x, y, z.

Let $\Gamma = \{a_1, a_2, \ldots, a_n, a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}\}$ be a generating set of a free group F_Γ. The elements of F_Γ are all *reduced* words over Γ, i.e., words not containing $a_ia_i^{-1}$ or $a_i^{-1}a_i$ as a subword. In this context, we call Γ a finite *group alphabet*, i.e., an alphabet with an involution. The multiplication of two elements (reduced words) $u, v \in F_\Gamma$ corresponds to the unique reduced word of the concatenation uv. This multiplication is called *concatenation* throughout the paper. Later in the