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The treewidth of proofs ☆

Moritz Müller [a,*], Stefan Szeider [b]

[a] Universität Wien, Kurt Gödel Research Center for Mathematical Logic, Währinger Straße 25, 1090 Vienna, Austria
[b] Technische Universität Wien, Algorithms and Complexity Group, Favoritenstraße 9-11, 1040 Vienna, Austria

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ABSTRACT

So-called *ordered* variants of the classical notions of pathwidth and treewidth are introduced and proposed as proof theoretically meaningful complexity measures for the directed acyclic graphs underlying proofs. Ordered pathwidth is roughly the same as proof space and the ordered treewidth of a proof is meant to serve as a measure of how far it is from being treelike. Length-space lower bounds for k-DNF refutations are generalized to arbitrary infinity axioms and strengthened in that the space measure is relaxed to ordered treewidth.

1. Introduction

Razborov says that "in most cases the basic question of propositional proof complexity boils down to this. Given a mathematical statement encoded as a propositional tautology ϕ and a class of admissible mathematical proofs formalized as a propositional proof system P, what is the minimal possible complexity of a ϕ-proof of ϕ?" [41, p. 415]. This is also the perspective of "Bounded Reverse Mathematics" taken in Cook and Nguyen's monograph [13, p. xiv].

1.1. Resolution-based proof systems

A proof system of fundamental interest is Resolution. The most important complexity measures for refutations are the *length*, the *width* and the *space* of a resolution refutation. Space (formula-space or clause-space) has been introduced by Esteban and Torán [18]. Intuitively, a space 100 refutation of a set Γ of clauses is one that can be presented as follows.

> A teacher is in class equipped with a blackboard containing up to 100 clauses. The teacher starts from the empty blackboard and finally arrives at one containing the empty clause. The blackboard can be altered by either writing down a clause from Γ, or by wiping out some clause, or by deriving a new clause from clauses currently written on the blackboard by means of the Resolution rule.

Some interesting restrictions of Resolution are obtained by requiring a particular simple structure of the DAGs (directed acyclic graphs) underlying refutations. Examples are *Input*, *linear* and *treelike* Resolution – we refer to the monograph [26].

☆ An extended abstract of this work appeared as [35].

* Corresponding author.
E-mail addresses: moritz.mueller@univie.ac.at (M. Müller), stefan@szeider.net (S. Szeider).

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Interesting extensions of Resolution include $R(1), R(2), \ldots, R(\log)$ from [28]. The system $R(1)$ is just Resolution, and $R(k)$ is a straightforward generalization operating with k-DNFs instead of clauses. The treelike versions of these systems are all simulated by (daglike) Resolution [27], so all treelike and daglike systems $R(k)$ line up in a hierarchy. The hierarchy is strict with respect to length as shown in [17] for the treelike systems and in [46,44] for the daglike ones. The hierarchy is also strict with respect to space, see again [17] for the treelike, and [7] for the daglike systems.

From a practical perspective the special interest in Resolution derives from its connections to SAT-solvers with length and space of refutations corresponding to time and space of algorithms. We refer to [37] for a recent survey. From the more theoretical perspective of "Bounded Reverse Mathematics", the systems $R(k)$ deserve some special interest because length lower bounds for them imply independence from weak arithmetics based on various forms of $\forall\exists$-induction schemes. For example, super-quasipolynomial length lower bounds on treelike or daglike $R(\log)$ imply independence from relativized bounded arithmetics $T^1_2(\alpha)$ or $T^2_2(\alpha)$ respectively [28]. See [9] for independence derivable from super-polynomial length lower bounds for daglike $R(1), R(2), \ldots$

Concerning the relationship of the complexity measures for (daglike) Resolution, Ben-Sasson and Wigderson [8] famously showed how to derive length lower bounds from width lower bounds. Also space lower bounds follow from width lower bounds [2] (see [19] for a recent alternative proof) but not vice-versa [36]. Ben-Sasson [6] initiated "the research of optimizing two of the measures at once" [6] and proved a *trade-off*, i.e. a negative answer, for length and width in treelike Resolution. Recently, Razborov [42] found an "ultimate" such trade-off. Ben-Sasson and Nordström [7] proved various trade-offs for length and space, for example, they constructed CNFs refutable by (daglike) Resolution in length $O(n)$ as well as in space $O(n/\log n)$, but every refutation in this space has length $2^{n^{\Omega(1)}}$. Beame et al. [5] found a length-space trade-off applying to Resolution refutations of superlinear space.

1.2. Infinity axioms

Many of the abovementioned lower bounds for the different complexity measures are witnessed by quite artificial CNFs. Recalling the introductory quote, CNFs that naturally express certain combinatorial principles deserve some special interest. A large class of such CNFs is obtained from first-order sentences φ letting CNFs $\langle\varphi\rangle_n$ naturally describe models of φ of size n. If φ does not have finite models, then these CNFs are contradictory and we ask for the complexity to refute them. If φ has no model at all, there are polynomial length refutation even in treelike Resolution [43]. If φ has no finite but an infinite model, i.e., φ is an *infinity axiom*, then exponential length lower bounds have been shown for the treelike systems, namely $2^{\Omega(n)}$ for treelike Resolution by Riis [43], $2^{\Omega(n\log k/k)}$ for treelike $R(k)$ by Dantchev and Riis [16], and already earlier $2^{\Omega(\sqrt{n})}$ for treelike $R(\log)$ by Krajíček [29].

But the daglike systems have short refutations of some infinity axioms. Stålmarck [47] gave a polynomial length Resolution refutation of the (CNFs expressing the) *least number principle*, the infinity axiom asserting a pre-order without minimal elements. Dantchev and Riis [16] showed that Resolution needs exponential length to refute any *relativized* infinity axiom. Iterating relativizations of the least number principle yields natural witnesses to the exponential separations of $R(k)$ and $R(k+1)$ [15]. It is not understood which (say, by some model-theoretic criterion) infinity axioms do have short refutations, say, in $R(k)$ for constant k; see [14] for a discussion.

As a second example, Maciel et al. [32] gave quasipolynomial length $R(\log)$-refutations of the *weak pigeonhole principle* with n^2 pigeons and n holes. It is not known whether this can be improved to polynomial. A lower bound $2^{\Omega(n/(\log n)^2)}$ is known [40] for Resolution. We refer to [39,45] for surveys of the proof complexity of pigeonhole principles.

For Resolution, space lower bounds have been obtained in [18] for the pigeonhole principles and in [1] for the least number principle. [17] generalizes these bounds to $R(k)$.

1.3. Ordered treewidth

Short $R(\log)$-refutations of infinity axioms cannot be treelike, in Razborov's words, they "must necessarily use a high degree of parallelism." [42, Abstract]. It would be desirable to quantify the amount of parallelism used by a proof and consider it as a complexity measure of proofs.

An hint how to do so comes from considering space. Space can be seen as a connectivity measure of the DAG underlying a refutation: Esteban and Torán [18] characterized space as a certain pebbling number of the refutation DAG. Following Beame et al. [5] the space of a linearly written Resolution refutation is the minimal number w such that at any derivation step at most w many already derived clauses are to be used at a later step. These characterizations are superficially reminiscent of characterizations of pathwidth for undirected graphs (see [24] and [25]), the second being akin to the vertex separation number.

Pathwidth and treewidth play an important role in Robertson and Seymour's graph minors project and have evolved as very successful and ubiquitously used complexity measures of graphs. We refer to [10] for a survey. Many graph problems can be efficiently solved by dynamic programing on a tree-decomposition witnessing small treewidth (see e.g. [20, Chapter 11]), and in fact treewidth turned out to be the key parameter to understand the complexity of graph homomorphism problems ([22,33,31] is a sample of some seminal results).

With an eye to proof DAGs, we introduce notions of path and tree decompositions of digraphs with associated width notions *ordered pathwidth* and *ordered treewidth*. Starting with [23] a number of width notions for digraphs have been