Information and Computation 254 (2017) 41–58

Contents lists available at ScienceDirect

Information and Computation

www.elsevier.com/locate/yinco

# Ordinal recursive complexity of Unordered Data Nets ☆

Fernando Rosa-Velardo

*Dpto. de Sistemas Informáticos y Computación, Facultad de Informática, Universidad Complutense de Madrid, C/Profesor José García Santesmases, 9, 28040 Madrid, Spain*

ARTICLE INFO

*Article history:*
Received 27 November 2015
Received in revised form 24 November 2016
Available online 27 February 2017

*Keywords:*
Data nets
Ordinal recursive complexity
Hyper-Ackermannian problems
Fast-growing complexity hierarchy
Multisets over $\mathbb{N}^k$

ABSTRACT

Data Nets are a version of colored Petri nets in which tokens carry data from an infinite and linearly ordered domain. This is a very expressive class, though coverability and termination remain decidable. Those problems have recently been proven complete for the $\mathbf{F}_{\omega^{\omega^\omega}}$ class in the fast growing complexity hierarchy. We characterize the exact complexity of Unordered Data Nets (*UDN*), a subclass of Data Nets with *unordered* data. We bound the length of bad sequences in well-quasi orderings of multisets over tuples of naturals by adapting the analogous result by Schmitz and Schnoebelen for words over a finite alphabet. These bounds imply that both problems are in $\mathbf{F}_{\omega^\omega}$. We prove that this result is tight by constructing *UDN* that weakly compute fast-growing functions and their inverses. This is the first complete problem for $\mathbf{F}_{\omega^\omega}$ with an underlying wqo not based on finite words over a finite alphabet.



## 1. Introduction

Higman's Lemma [12] is a well-known result that states that whenever $(X, \leq)$ is a well-quasi order (wqo) then the embedding order $(X^*, \leq^*)$ in the set $X^*$ of finite words over $X$ is also a wqo. As a consequence, and because $\leq^*$ is a refinement of the multiset order $\leq^\oplus$ ($s \leq^* s'$ implies $s \leq^\oplus s'$), the order $\leq^\oplus$ over the set of multisets $X^\oplus$ is also a wqo.

However, multisets are intuitively a simpler domain than words. This is witnessed by their *maximal ordinal types* [14,20], which can be seen as a measure of their size. Indeed, if the order type of $X$ is $\alpha$ then the order type of $X^*$ is $\omega^{\omega^\alpha}$ [14], while the order type of $X^\oplus$ is only $\omega^\alpha$ [24].[1]

The ordinal type of a wqo has recently been used in several works [6,21,8,11,16] to characterize the ordinal-recursive complexity of (the verification of several problems for) *monotonic* systems over an underlying wqo, which have been called Well-Structured Transition Systems (WSTS) [2,9]. Prominent examples of such WSTS are Petri nets/VASS, affine nets [10], Lossy Channel Systems [1,5] or Data Nets [15]. Lossy Channel Systems (LCS) can be seen as finite state machines communicating over FIFO unreliable channels. Hence, if $\Gamma$ is the (finite) alphabet of messages, the state space is given by $Q \times (\Gamma^*)^k$ for some finite set of states $Q$ and $k \geq 0$.

Data Nets [15] are a very general (but monotonic) extension of Petri nets in which tokens are taken from a linearly ordered and dense domain, and whole-place operations like transfers or resets are allowed. They can be seen as arrays or lists of Petri nets (with whole-place operations) communicating by rendezvous and broadcasts [4]. Hence, the state space of a Data Net is given by some $(\mathbb{N}^k)^*$. It was shown in [3] that Petri Data Nets (Data Nets in which no whole-place operations

☆ Author is partially supported by the Spanish projects TRACES TIN2015-67522-C3-3-R and N-Greens S2013/ICE-2731.
*E-mail address:* fernandorosa@sip.ucm.es.

[1] This holds only for $\omega \leq \alpha < \epsilon_0$, which will be the case in the paper.

http://dx.doi.org/10.1016/j.ic.2017.02.002
0890-5401/

or broadcasts are allowed) are already as expressive as Data Nets. Data Nets are used e.g. in [7], to give semantics and prove decidability results of a very expressive formal model of asynchronous event-driven programs, allowing dynamic creation of concurrent tasks, events and threads.

A natural subclass of Data Nets is that of Unordered Data Nets (*UDN*), that is, the subclass of Data Nets in which nets are not in an array of nets, but in a pool of nets. Unlike in Data nets, where the underlying state space is based on finite words of tuples of naturals, in the case of Unordered Data Nets the state space is based on finite multisets of such tuples. Thus, in the unordered case the state space can be given by some $(\mathbb{N}^k)^{\oplus}$. Unordered Data Nets are very closely related to the class of so-called $\nu$-Petri nets. It can be seen as a restriction of *UDN* without broadcasts, so that each transition affects only a constant number of names given by the transition. In $\nu$-Petri nets tokens carry pure identifiers, that can only be compared with each other by equality, and can be created fresh (different from all other existing identifiers), which can be used for the modeling of distributed authentication protocols (see for instance [17]). Also, $\nu$-Petri nets and its polyadic version (in which tokens are tuples of identifiers) are in turn closely related to process algebra with name creation (e.g., channel names in the $\pi$-calculus) [19].

On the one hand, $\nu$-Petri Nets (*UDN* without whole-place operations or broadcasts) have been recently proven complete for the $\mathbf{F}_{\omega\cdot 2}$ class of double Ackermannian problems in the fast-growing complexity hierarchy [16]. On the other hand, (full) Data Nets have been recently proven to be complete for the $\mathbf{F}_{\omega^{\omega^\omega}}$ class in the fast-growing complexity hierarchy [11]. We fill part of the gap in between by proving that *UDN* are complete for $\mathbf{F}_{\omega^\omega}$.

Our proof relies on the techniques developed by Schnoebelen and Schmitz, both for the upper bounds as for the hardness result. For the upper bounds, we adapt the techniques in [21] to bound the length of (controlled) bad sequences in $(\mathbb{N}^k)^{\oplus}$, that is, sequences $s_0, ..., s_k$ such that $s_i \not\leq s_j$ for all $i < j$. These upper bounds give us ordinal recursive upper bounds for the coverability and termination problems. This result is general, and could therefore be used for other monotonic systems with a state space based on multisets of tuples of naturals. For the lower bound, we show (i) how we can encode ordinals below $\omega^{\omega^\omega}$ as markings of *UDN* and (ii) how we can use this encoding to perform weak computations of fast-growing functions and their inverses. This entails the corresponding lower bounds, by using the device presented for instance in [23].

Thus, the complexity of *UDN* sits at the exact same level as LCS, which were proven to be $\mathbf{F}_{\omega^\omega}$-complete in [6,21]. Let us remark that, because the state spaces of *UDN* ($(\mathbb{N}^k)^{\oplus}$) and the state spaces of LCS ($Q \times (\Gamma^*)^k$) are very different (though with comparable order types), it does not seem possible to perform direct reductions from one model to the other, which would yield alternative (and perhaps more direct) proofs of our results. We leave these reductions as open problems.

Finally, let us comment that the construction reducing Data nets to Petri Data Nets (removing broadcasts) [3] is no longer correct in the case of UDN. We will see that our hardness result heavily relies on broadcast operations that, for instance, empty a given place in all tuples. This is not a coincidence, since *UDN* (hyper Ackermannian) are strictly harder than $\nu$-Petri Nets (double Ackermannian).

The rest of the paper is structured as follows. Section 2 presents some definitions, notations and results we use in the paper. In Sect. 3 we obtain upper bounds for the length of controlled bad sequences in multiset wpos. In Sect. 4 we define *UDN*and obtain an upper bound for their coverability and termination problems. In Sect. 5 we consider lower bounds. Sect. 6 presents our conclusions and some open problems.

## 2. Preliminaries

**Well orders.** $(X, \leq_X)$ is a *quasi-order* (qo) if $\leq_X$ is a reflexive and transitive binary relation on $X$. For a qo we write $x <_X y$ iff $x \leq_X y$ and $y \not\leq_X x$. A *partial order* (po) is an antisymmetric quasi-order. A po $(X, \leq)$ is *total* (or *linear*) if for any $x, x' \in X$ either $x \leq x'$ or $x' \leq x$. We will shorten $(X, \leq_X)$ to $X$ when the underlying order is obvious. Similarly, $\leq$ will be used instead of $\leq_X$ when $X$ can be deduced from the context.

We say a (finite or infinite) sequence $(x_i)_{i\leq\omega}$ is *good* if there are indices $i < j$ such that $x_i \leq x_j$. Otherwise, we say it is *bad*. A po $X$ is a *well partial order* (wpo) if every bad sequence is finite.

If $X_1$ and $X_2$ are wpos, their Cartesian product $X_1 \times X_2$ is well ordered by $(x_1, x_2) \leq_{X_1\times X_2} (x'_1, x'_2)$ iff $x_1 \leq_{X_1} x'_1$ and $x_2 \leq_{X_2} x'_2$. Their disjoint sum $X_1 + X_2 = (\{1\} \times X_1) \cup (\{2\} \times X_2)$ is well partially ordered by $\langle i, x\rangle \leq_{X_1+X_2} \langle j, x'\rangle$ iff $i = j$ and $x \leq_{X_i} x'$.

**Functions.** If $X$ and $Y$ are ordered, a mapping $f : X \to Y$ is *increasing* (resp. *strictly increasing*) if $x \leq_X y$ implies $f(x) \leq_Y f(y)$ (resp. if $x <_X y$ implies $f(x) <_Y f(y)$); $f$ is an *order embedding* (shortly: embedding) if $f(x) \leq_Y f(x')$ iff $x \leq_X x'$. A bijective order embedding is called an *order isomorphism* (shortly: isomorphism). Two po $X$ and $Y$ are isomorphic if there is an isomorphism between them, in which case we write $X \equiv Y$.

**Multisets.** Given a set $X$, we denote by $X^{\oplus}$ the set of finite multisets of $X$, that is, the set of mappings $m : X \to \mathbb{N}$ with a finite support $sup(m) = \{x \in X \mid m(x) \neq 0\}$. We use the set-like notation for multisets when convenient, with $\{x^n\}$ describing the multiset with $n$ occurrences of $x$. We use $+$ and $-$ for multiset addition and subtraction, respectively defined by $(m + m')(x) = m(x) + m'(x)$ and $(m - m')(x) = max(m(x) - m'(x), 0)$. If $X$ is a wpo then so is $X^{\oplus}$ ordered by $\leq_{\oplus}$ defined by $\{x_1, \ldots, x_n\} \leq_{\oplus} \{x'_1, \ldots, x'_m\}$ if there is an injection $h : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $x_i \leq_X x'_{h(i)}$ for each $i \in \{1, \ldots, n\}$.

**Words.** Given a set $X$, any $u = x_1 \cdots x_n$ with $n \geq 0$ and $x_i \in X$, for all $i \in \{1, ..., n\}$, is a finite word on $X$. We denote by $X^*$ the set of finite words on $X$. If $n = 0$ then $u$ is the empty word, which is denoted by $\varepsilon$. If $X$ is a wpo then so is $X^*$ ordered by $\leq_{X^*}$ which is defined as follows: $x_1 \ldots x_n \leq_{X^*} x'_1 \ldots x'_m$ if there is a strictly increasing mapping $h : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $x_i \leq_X x'_{h(i)}$ for each $i \in \{1, \ldots, n\}$ (Higman's Lemma). This is called the *embedding order*, in contrast with the