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Enlarging learnable classes

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ABSTRACT

We study which classes of recursive functions satisfy that their union with any other explanatorily learnable class of recursive functions is again explanatorily learnable. We provide sufficient criteria for classes of recursive functions to satisfy this property and also investigate its effective variants. Furthermore, we study the question which learners can be effectively extended to learn a larger class of functions. We solve an open problem by showing that there is no effective procedure which does this task on all learners which do not learn a dense class of recursive functions. However, we show that there are two effective extension procedures such that each learner is extended by one of them.

1. Introduction

One branch of inductive inference investigates the learnability of recursive functions; the basic scenario given in the seminal paper by Gold [10] is as follows. Let $\mathcal{S}$ be a class of recursive functions; we say that $\mathcal{S}$ is *explanatorily learnable* iff there is a learner M which issues conjectures $e_0, e_1, \ldots$ with e_n being based on the data $f(0), f(1), \ldots, f(n-1)$ (e_0 being based on no data) such that, for all $f \in \mathcal{S}$, almost all of these conjectures are the same index e explaining f, that is, satisfying $\varphi_e = f$ with respect to an underlying numbering $\varphi_0, \varphi_1, \ldots$ of all partial recursive functions. In this paper, we consider learnability by partial recursive learners; with M_e we refer to the learner derived from the e-th partial recursive function. This setting of learning is also called *learning in the limit* (see also [20], which surveys recursive function learning).

During the course of time, several variants of this basic notion of explanatory learning (**Ex**) have been considered; most notably, *behaviourally correct learning* (**BC**) [3], in which the learner has to almost always output a correct index for the input function (these indices though are not constrained to be the same).

Another variant considered is *finite learning* (**Fin**) where the learner outputs a special symbol (?) until it makes one conjecture e which is never abandoned; this conjecture must of course be correct for a function to be learnt. Osherson, Stob and Weinstein [16] introduced a generalisation of this notion, namely *confident learning* (**Conf**), where the learner can revise the hypothesis finitely often; it must, however, on each total function f, even if it is not in the class to be learnt or not even recursive, eventually stabilise on one conjecture e. In inductive inference, one often only needs the weak version

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of this property where the convergence criterion only applies to recursive functions while the convergence behaviour on non-recursive ones is not constrained (**WConf**, Sharma, Stephan and Ventsov [19]).

Minicozzi [15] called a learner *reliable* (**Rel**) iff the learner, on every function (including total but non-recursive functions), either converges to a correct index or signals infinitely often that it does not find the index (by doing a mind change or outputting a question mark); Osherson, Stob and Weinstein [16, Exercise 4.6.1A] generalised this notion to *weakly reliable* (**WRel**) where the learner is weakly reliable iff the learner is reliable on every recursive function and there are no constraints on its behaviour on non-recursive functions. One can combine the notion of reliability and confidence: A learner is *weakly confident and weakly reliable* (**WConfRel**) iff the learner, for every recursive function f, either converges to an index e with $\varphi_e = f$ or almost always outputs ? (in order to signal non-convergence to any conjecture).

Formal definitions of the above criteria are given in Section 2. The relations between those criteria have been extensively studied, giving the following inclusion relations [4,6,8,10,12,15,16,19]:

- **Fin** $\subset$ **Conf** $\subset$ **WConf** $\subset$ **Ex** $\subset$ **BC**;
- **ConfRel** $\subset$ **WConfRel** $\subset$ **WRel** $\subset$ **Ex** $\subset$ **BC**;
- **Rel** $\subset$ **WRel** $\subset$ **Ex** $\subset$ **BC**;
- **Fin** $\not\subset$ **Rel** and **Rel** $\not\subset$ **WConf**.

Besides inclusion (learnability with respect to which criterion implies learnability with respect to another criterion), structural questions have also been studied: Is the union of two learnable classes learnable? Can one extend each learnable class?

Bārzdiņš [3] and Blum and Blum [4] gave with the *non-union theorem* a quite strong answer to the first question: There are two classes $\mathcal{S}$ and $\mathcal{S}'$ of recursive functions such that each of them is learnable under the criterion **Ex** but their union is not learnable even under the more general criterion **BC**. Indeed, one can even learn the class $\mathcal{S}$ confidently and the class $\mathcal{S}'$ reliably. Thus, the non-union theorem gives an interesting contrast to the fact that both confident learning and reliable learning are effectively closed under union, that is, given two confident (reliable) learners, one can effectively find a confident (reliable) learner which explanatorily learns the union of functions explanatorily learnt by the two learners. In fact Minicozzi [15] proved a stronger result that, given an index for a recursively enumerable set A of reliable learners, one can effectively find a reliable learner which learns the union of the classes of functions learnt by the individual learners in the set A. Apsītis, Freivalds, Simanovskis and Smotrovs [2] also considered closedness properties of **Ex**-identification.

The non-union theorem has been extended in various ways; for example, if one considers learning with oracles, there is a choice of a confidently learnable $\mathcal{S}$ and a consistently learnable $\mathcal{S}'$ such that their union is not **Ex**-learnable, even with any non-high oracle A [9,13] — "non-high" is the best that one can expect in this context as a high oracle permits to learn the whole class of recursive functions [1].

Furthermore, it is interesting to ask how effective the union is. That is, if the union of two classes is learnable, can one effectively construct a learner for the union, given programs for the learners of the two given classes? The answer is "No" in general as can be seen directly by the proof of the non-union theorem.

The confidently learnable class $\mathcal{S}$ above consists of all the functions f such that $f(0)$ is an index for f, and the class $\mathcal{S}'$ consists of all the functions f which are almost everywhere 0 (Blum and Blum [4] used slightly different classes $\mathcal{S}$ and $\mathcal{S}'$ which were $\{0,1\}$-valued; our $\mathcal{S}$ and $\mathcal{S}'$, which are taken from [3], make the presentation simpler; a proof of the non-union theorem using these classes can also be found in [20]). Now consider the union of $\mathcal{S}'$ with a class $\mathcal{S}_e$, where $\mathcal{S}_e$ contains φ_e in the case that φ_e is total and $\varphi_e(0) = e$; otherwise $\mathcal{S}_e$ is empty. It is easy to show that, for each e, the class $\mathcal{S}_e \cup \mathcal{S}'$ is explanatory (**Ex**) learnable. However, learnability of these unions is not effective. If, given e, one can effectively find a **Ex**-learner $M_{h(e)}$ for the class $\mathcal{S}_e \cup \mathcal{S}'$, then one could make a learner N for $\mathcal{S} \cup \mathcal{S}'$ as follows. For non-empty sequences σ, $N(\sigma) = M_{h(\sigma(0))}(\sigma)$. This learner N, **Ex**-learns $\mathcal{S} \cup \mathcal{S}'$, as $\sigma(0)$ constrains the functions in $\mathcal{S}$ to be only from $\mathcal{S}_{\sigma(0)}$. However, this is in contradiction to the non-union theorem, and thus one cannot effectively find, given e, an **Ex**-learner for $\mathcal{S}_e \cup \mathcal{S}'$.

The above example suggests to study four notions of when the unions of a given class $\mathcal{S}$ with another class is **Ex**-learnable:

1. $\mathcal{S}$ is (non-constructively) **Ex**-*unionable* iff for every **Ex**-learnable class $\mathcal{S}'$, the class $\mathcal{S} \cup \mathcal{S}'$ is **Ex**-learnable;
2. $\mathcal{S}$ is *constructively* **Ex**-*unionable* iff one can effectively convert every **Ex**-learner for a class $\mathcal{S}'$ into an **Ex**-learner for the class $\mathcal{S} \cup \mathcal{S}'$;
3. $\mathcal{S}$ is *singleton*-**Ex**-*unionable* iff for every recursive g, $\mathcal{S} \cup \{g\}$ is **Ex**-learnable;
4. $\mathcal{S}$ is *constructively singleton*-**Ex**-*unionable* iff there is a recursive function which assigns, to every index e, an **Ex**-learner for the class $\mathcal{S} \cup \{\varphi_e\}$ if φ_e is total and for the class $\mathcal{S}$ if φ_e is partial.

The same notions can also be defined for other learning criteria like finite, confident and behaviourally correct learning. We get the following results:

1. If a class $\mathcal{S}$ has a weakly confident learner then it is constructively singleton-**Ex**-unionable.
2. If a class $\mathcal{S}$ has a weakly confident and weakly reliable learner then it is constructively **Ex**-unionable.