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Lower bounds on the redundancy in computations from random oracles via betting strategies with restricted wagers ☆

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ABSTRACT

The Kučera–Gács theorem is a landmark result in algorithmic randomness asserting that every real is computable from a Martin-Löf random real. If the computation of the first n bits of a sequence requires $n + h(n)$ bits of the random oracle, then h is the *redundancy* of the computation. Kučera implicitly achieved redundancy $n \log n$ while Gács used a more elaborate coding procedure which achieves redundancy $\sqrt{n} \log n$. A similar bound is implicit in the later proof by Merkle and Mihailović. In this paper we obtain optimal strict lower bounds on the redundancy in computations from Martin-Löf random oracles. We show that any nondecreasing computable function g such that $\sum_n 2^{-g(n)} = \infty$ is not a general upper bound on the redundancy in computations from Martin-Löf random oracles. In fact, there exists a real X such that the redundancy g of any computation of X from a Martin-Löf random oracle satisfies $\sum_n 2^{-g(n)} < \infty$. Moreover, the class of such reals is comeager and includes a Δ^0_2 real as well as all weakly 2-generic reals. On the other hand, it has been recently shown that any real is computable from a Martin-Löf random oracle with redundancy g, provided that g is a computable nondecreasing function such that $\sum_n 2^{-g(n)} < \infty$. Hence our lower bound is optimal, and excludes many slow growing functions such as $\log n$ from bounding the redundancy in computations from random oracles for a large class of reals. Our results are obtained as an application of a theory of effective betting strategies with restricted wagers which we develop.

1. Introduction

Every sequence is computable from a sequence which is random in the sense of Martin-Löf [5]. This major result in algorithmic information theory is known as the Kučera–Gács theorem and was proved by Kučera [1,6] and Gács [2]. Both authors showed that the use of the oracle in these reductions can be bounded above by a computable function, but Kučera did not focus on minimizing the number of bits of the oracle that are needed to compute the first n bits of the sequence.

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If the latter number is $n + h(n)$, we say that the computation has *redundancy* h. A close look at Kučera's argument shows that his techniques achieve redundancy $n \log n$. Gács, on the other hand, took special care to minimize the oracle use. His argument produces a slightly more elaborate computation with redundancy $3\sqrt{n}\log n$, which can easily be improved to $\sqrt{n}\log n$. Both of the arguments were formulated in terms of effective measure, i.e. according to the Martin-Löf definition of randomness.

The major difference between the results of Kučera and Gács is that the latter provides a reduction with oracle use $n + \mathbf{o}(n)$ while the former does not. Merkle and Mihailović [3] presented a proof in terms of effective martingales, using similar ideas to Gács' proof but expressed in terms of betting strategies. Up to now, the only known strict lower bound on the redundancy in computation from Martin-Löf random reals is the constant bound, and is due to Downey and Hirschfeldt [7, Theorem 9.13.2]. Turing reductions with constant redundancy are also known as *computably Lipschitz* or *cl* reductions and are well studied in computability theory, e.g. see [7, Chapter 9]. Downey and Hirschfeldt showed that the redundancy in the Kučera–Gács theorem cannot be $\mathbf{O}(1)$. In fact, they constructed a sequence which is not computed with constant redundancy by any real whose Kolmogorov complexity is bounded below by a computable nondecreasing unbounded function. The reals with the latter property are sometimes known as *complex reals*. A close look at this argument reveals that the set of reals which cannot be computed from any complex real with constant redundancy is comeager. Moreover, it follows from the effective nature of the argument that:

a weakly 2-generic real cannot be computed by any complex real with constant redundancy,

where a real is called *weakly 2-generic* if it has a prefix in every dense Σ^0_2 set of strings.

By [8] a real which is not complex has infinitely many initial segments of trivial complexity in the sense that $C(X\restriction_n) = C(n) + \mathbf{O}(1)$ and $K(X\restriction_n) = K(n) + \mathbf{O}(1)$, where K and C denote the prefix-free and plain Kolmogorov complexities. Sequences with the latter property are known as *infinitely often C-trivial and K-trivial* respectively. It follows that any sequence computing a weakly 2-generic sequence with constant redundancy is infinitely often C-trivial and infinitely often K-trivial.

1.1. Our results, in context

In Section 3 we show that the redundancy in computations from Martin-Löf random oracles cannot be bounded by certain slow growing functions. Recall that a real is Δ^0_2 if and only if it is computable from the halting problem.

Theorem 1.1. *There exists a real X such that $\sum_i 2^{-g(i)} < \infty$ for every nondecreasing computable function g for which there exists a Martin-Löf random real Y which computes X with redundancy g. In fact, the reals X with this property form a comeager class which includes every weakly 2-generic real.*

This result implies that any nondecreasing computable function g such that $\sum_i 2^{-g(i)} = \infty$ is not a general upper bound on the redundancy in computations of reals from Martin-Löf random oracles. A typical function with this property is $\lceil \log n \rceil$, so the Kučera–Gács theorem does not hold with redundancy $\lceil \log n \rceil$. On the other hand, if $g(n) = 2 \cdot \lceil \log n \rceil$ then $\sum_i 2^{-g(i)} < \infty$. It was recently shown in [4] that any nondecreasing computable function g with the latter property is a general upper bound on the redundancy in computations of reals from Martin-Löf random oracles. Hence Theorem 1.1 is optimal and gives a characterization of the computable nondecreasing redundancy upper bounds in computations of reals from Martin-Löf random oracles. Note that the optimal bounds obtained in [4] are exponentially smaller than the previously best known upper bound of $\sqrt{n}\log n$ from Gács [2].

With slightly more effort we also obtain an effective version of Theorem 1.1, which gives many more examples of reals X which can only be computed from random oracles with large redundancy. Recall that the halting problem relative to A is denoted A'. The generalized non-low$_2$ reals are an important and extensively studied class in the context of degree theory: A is generalized low$_2$ if A'' has the same Turing degree as $(A \oplus \emptyset')'$, and a set which is not generalized low$_2$ is called generalized non-low$_2$.

Theorem 1.2 *(Jump hierarchy). Every set which is generalized non-low$_2$ (including the halting problem) computes a real X with the properties of Theorem 1.1.*

The proof of Theorem 1.1 also gives a nonuniform version of the latter result, requiring a weaker condition regarding the computational power of the oracle. Recall from [9] that a set A is array noncomputable if for each function f that is computable from the halting problem with a computable upper bound on the oracle use, there exists a function h which is computable from A and which is not dominated by f. A degree is array noncomputable if its members are. The class of array noncomputable degrees (again an extensively studied class) is an upwards closed superclass of the generalized non-low$_2$ degrees, and includes low degrees amongst its members.

Theorem 1.3 *(Array noncomputability). Suppose that $\sum_i 2^{-g(i)} = \infty$ for some computable nondecreasing function g. Then every array noncomputable real computes a real X which is not computable by any Martin-Löf real with redundancy g.*

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