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Approximating weighted neighborhood independent sets

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ABSTRACT

A neighborhood independent set (NI-set) is a subset of edges in a graph such that the closed neighborhood of any vertex contains at most one edge of the subset. Finding a maximum cardinality NI-set is an NP-complete problem. We consider the weighted version of this problem. For general graphs we give an algorithm with approximation ratio Δ, and for diamond-free graphs we give a ratio $\Delta/2+1$, where Δ is the maximum degree of the input graph. Furthermore, we show that the problem is polynomially solvable on cographs. Finally, we give a tight upper bound on the cardinality of a NI-set on regular graphs.

1. Introduction

Let $G=(V,E)$ be a simple undirected graph. For $v \in V$, let $N(v)=\{u : uv \in E\}$ be the open neighborhood of v; and let $N[v]=N(v)\cup\{v\}$ be the closed neighborhood of v. Let $\Delta(G)=\max_{v\in V}|N(v)|$ be the maximum degree of a vertex in G, if the context is clear we write $\Delta=\Delta(G)$. A subset $S\subseteq E$ is *neighborhood independent* if $|E[v]\cap S|\le 1$ for any vertex $v\in V$, where $E[v]$ denotes the set of edges in the subgraph induced by $N[v]$. The goal of the maximum NI-set problem is to find a NI-set S of maximum cardinality. The decision version of the problem is formulated as follows: given an integer k and a graph G, decide whether G contains a NI-set of size at least k.

In 1986, Lehel and Tuza [1] gave a linear time algorithm for interval graphs. Wu [2] gave a $O(n^3)$ algorithm for strongly chordal graphs. Tuza et al. [3] proved the problem to be NP-complete on split graphs whose vertices of the independent set have degree 3; and gave a linear time algorithm for strongly chordal graphs if a strong elimination order is given as input. Guruswami and Rangan [4] proved the problem to be NP-complete for diamond-free planar graphs with $\Delta=3$. In the same work it is shown that the problem is NP-complete on line graphs with $\Delta=3$. Warnes [5] gave a linear time algorithm for tree-cographs and P_4-tidy graphs, and proved the problem to be NP-complete on co-bipartite graphs. Other non-algorithmic results related to this problem can be found in [6,7]. A natural generalization of the NI-set problem was considered in [8]. We remark that the above mentioned results are for the unweighted version of the problem. To our best knowledge, no approximation algorithms for this problem were explored before.

In this work we consider the weighted version of this problem, which we call *Maximum Weighted NI-set (MWNI)*. Formally, given an edge-weighted graph we are to find a NI-set that maximizes its total weight. To our best knowledge, MWNI was not studied before. First we argue that this problem is hard to approximate. Then we show that a simple greedy algorithm yields an approximation ratio Δ for general graphs. Furthermore, we propose a fractional local ratio algorithm for diamond-free graphs with approximation ratio $\Delta/2+1$. We give a polynomial time algorithm for cographs. Finally, a tight bound on the cardinality of a NI-set on d-regular graphs is given.

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We close the introduction with some definitions and notation used in this paper. Let $dist_G(u, v)$ be the distance between the vertices u and v in G. A maximal complete subgraph $K = (V(K), E(K))$ of G with at least two vertices is a *clique* of G. Let $K_{i,j}$ be the complete bipartite graph of i vertices in one partition and j in the other. Given a vector x over E and a subset $F \subseteq E$, we define $x(F) = \sum_{f \in F} x_f$. For an edge $uv = e \in E$, let $C(e) \subseteq E$ be the set of edges which are *in conflict* with e; more formally, $C(e) = \bigcup_{w \in N[u] \cap N[v]} E[w]$. Note that $e \in C(e)$.

2. Inapproximability

Let us first briefly observe that the NI-set problem is hard to approximate within a ratio $O(\Delta^{1-\epsilon})$ for any $\epsilon > 0$. The implications of this are twofold: on the one hand, it rules out any constant approximation ratio; and on the other hand, it proves that our algorithms are in a sense tight.

Theorem 1. *For $\Delta \geq 4$, MWNI is NP-hard to approximate within a ratio*

$$\frac{\Delta - 1}{2^{O\left(\sqrt{\log(\Delta-1)}\right)}}.$$

Proof. To show this we give a reduction from independent set that preserves approximability and use a hardness result given by Trevisan in [9].

Let G be a graph with $\Delta(G) \geq 3$ and suppose we are to compute an independent set of G. Construct the graph H as follows. For each vertex $u \in V(G)$ add two adjacent vertices u and u' to H. For each edge $uv \in E(G)$ add a new vertex c_{uv} in H adjacent to u, u', v and v'. For an edge of type uu' we set $w_{uu'} = 1$; and $w_e = 0$ for any edge e incident to some c_{uv}. There is a direct correspondence between NI-sets of H with no zero-weight edges and independent sets in G preserving their sizes: for an independent set I of G we define the NI-set $\{uu' : u \in I\}$ in H; and for an NI-set S of H we define the independent set $\{u : uu' \in S\}$ in G. It is clear that a β-approximate NI-set of H amounts to a β-approximate independent set of G.

A key observation is that $\Delta(H) = \max\{\Delta(G) + 1, 4\} = \Delta(G) + 1$. Trevisan [9] observed that it is NP-hard to approximate the maximum independent set problem within a ratio $\Delta(G)/2^{O\left(\sqrt{\log \Delta(G)}\right)}$. Therefore, it is NP-hard to approximate MWNI within a ratio

$$\frac{\Delta(H) - 1}{2^{O\left(\sqrt{\log(\Delta(H)-1)}\right)}}. \quad \square$$

Remark 1. For any $\epsilon > 0$ and $\Delta \geq 4$, MWNI cannot be approximated within a ratio $O(\Delta^{1-\epsilon})$, unless P = NP.

Proof. This follows because for any constant $c > 0$ we have

$$\Delta^{1-\epsilon} = \frac{\Delta}{2^{\epsilon \log \Delta}} = o\left(\frac{\Delta - 1}{2^{c\sqrt{\log(\Delta-1)}}}\right). \quad \square$$

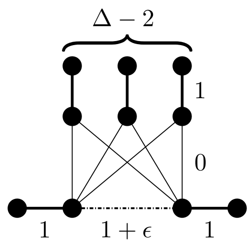

Fig. 1. Tightness of the greedy algorithm.

3. A Δ-approximation algorithm

If $\Delta \leq 2$, then computing a maximum weighted NI-set amounts to finding a maximum weighted matching; the set S is the maximum weighted matching. We thus consider graphs with $\Delta \geq 3$. We first introduce a technical result.

Lemma 2. *If S is a NI-set of G and $\Delta \geq 3$, then $|S \cap C(e)| \leq \Delta$ for each $uv = e \in E$.*

Proof. Let $W = N[u] \cap N[v] \setminus \{u, v\}$. Observe that $|W| \leq \Delta - 1$. If $|W| = \Delta - 1$, then $S \cap C(e) = S \cap \bigcup_{w \in W} E[w]$, and therefore $|S \cap C(e)| \leq \sum_{w \in W} |E[w] \cap S| \leq |W| = \Delta - 1$. If $|W| \leq \Delta - 2$, then $S \cap C(e) = S \cap (E[u] \cup E[v] \cup \bigcup_{w \in W} E[w])$, and therefore $|S \cap C(e)| \leq 2 + |W| \leq \Delta$. $\square$

Consider the natural greedy approach: begin with an empty solution S, and in each iteration add to S the edge $e \in E \setminus S$ of maximum weight such that $|(S \cup \{e\}) \cap E[v]| \leq 1$ for each $v \in V$. Return the constructed set once no more edges can be added.

Let S^* be an optimal NI-set of G. Each time we add an edge e to S, the edges in $C(e)$ cease to be candidates for future iterations—they become blocked. Among the edges in $C(e)$ some may have been blocked in a previous iteration while some in the current. Those in $S^* \cap C(e)$ blocked in a previous iteration are already accounted for by some $e' \in S$. And those in $S^* \cap C(e)$ blocked in the current iteration are at most Δ by the above lemma, and each of them has weight at most w_e by the choice of e. When the algorithm ends, all edges in S^* are blocked by S, and it follows that $w(S^*) \leq \Delta w(S)$. This implies an approximation ratio Δ.

Remark 2. The analysis of the algorithm is tight.

Proof. Consider the graph of Fig. 1. The greedy algorithm outputs the sole edge of weight $1 + \epsilon$, while the optimum corresponds to the bold edges of total weight Δ. Therefore, the ratio between the optimum and the computed solution is $\Delta/(1 + \epsilon)$. $\square$

4. A $(\Delta/2 + 1)$-approximation algorithm for diamond-free graphs

In this section we give a fractional local ratio approximation algorithm for the MWNI problem on diamond-free graphs. To the unfamiliarized reader, the intuition be-