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An implicit degree sum condition for cycles through specified vertices

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A B S T R A C T

In 1989, Zhu, Li and Deng introduced the concept of implicit degree. For a subset S of $V(G)$, let $i\Delta_2(S)$ denote the maximum value of the implicit degree sum of two vertices in S. In this paper, we prove that: Let G be a 2-connected graph on n vertices and X be a subset of $V(G)$. If $i\Delta_2(S) \geq n$ for each independent set S of order $\kappa(X)+1$ in $G[X]$, then G has a cycle containing all vertices of X. This result generalize one result of Yamashita (2008) [14].

1. Introduction

In this paper, we consider only finite, undirected and simple graphs. Notation and terminology not defined here can be found in [2].

Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $\alpha(G)$ and $\kappa(G)$ denote the *independence number* and the *connectivity* of G, respectively. For a vertex $u \in V(G)$ and a subgraph H of G, $N_H(u)$ and $d_H(u)$ denote the set and the number of vertices adjacent to u in H, respectively. We always call $N_H(u)$ and $d_H(u)$ the *neighborhood* and the *degree* of u in H, respectively. For two vertices $u, v \in V(G)$, $d_H(u,v)$ denotes the length of the shortest path connecting u and v in H. For $X \subset V(G)$, $N_H(X)=\cup_{x\in X} N_H(x)$. If $H=G$, we always use $N(u)$, $d(u)$, $d(u,v)$ and $N(X)$ in place of $N_G(u)$, $d_G(u)$ $d_G(u,v)$ and $N_G(X)$, respectively. Let $N_2(u)=\{v \in V(G) : d(u,v)=2\}$. For a nonempty set $S \subset V(G)$, let $\Delta_k(S)=\max\{\sum_{x\in X} d(x) :$ X is a subset of S with k vertices$\}$.

A graph G is called *hamiltonian* if it contains a hamiltonian cycle, i.e. a cycle that contains all vertices of G. The hamiltonian problem is an important problem in graph theory. Since hamiltonian problem is an NP-complete problem, various sufficient conditions for a graph to be hamiltonian have been given by researchers. The following two sufficient conditions are well-known.

Theorem 1 *([11]). Let G be a graph on $n \geq 3$ vertices. If $d(x)+d(y) \geq n$ for every pair of nonadjacent vertices x and y in G, then G is hamiltonian.*

Theorem 2 *([7]). Let G be a 2-connected graph. If $\alpha(G) \leq \kappa(G)$, then G is hamiltonian.*

Let X be a subset of $V(G)$. $G[X]$ denotes the subgraph of G induced by X. X is defined as *cyclable* in G if there exists a cycle in G containing all vertices of X. We also say that G is X-cyclable. A graph G being hamiltonian is equivalent to G being $V(G)$-cyclable. Hence we can regard hamiltonian problem as a special case of cyclable problem. Therefore the following theorem is a generalization of Theorem 1.

Theorem 3 *([13], [12]). Let G be a 2-connected graph on n vertices and X be a subset of $V(G)$. If $d(x)+d(y) \geq n$ for every pair of nonadjacent vertices x and y in $G[X]$, then G is X-cyclable.*

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In 1985, Fournier generalized Theorem 2 as follows.

Theorem 4 *([9]). Let G be a 2-connected graph and X be a subset of $V(G)$. If $\alpha(G[X]) \le \kappa(G)$, then G is X-cyclable.*

If $G[X]$ is not complete, we denote by $\kappa(X)$ the minimum cardinality of a set of vertices in G separating two vertices of X in G. Note that $\kappa(G) \le \kappa(X)$ for a graph G and $X \subseteq V(G)$. In 1997, Broersma et al. gave an improvement of Theorem 4 as follows.

Theorem 5 *([3]). Let G be a 2-connected graph and X be a subset of $V(G)$. If $\alpha(G[X]) \le \kappa(X)$, then G is X-cyclable.*

In 2005, Flandrin et al. gave another generalization of Theorem 1.

Theorem 6 *([8]). Let G be a k-connected graph on n vertices ($k \ge 2$), $X_1, X_2, \ldots, X_k$ be subsets of $V(G)$ and $X = X_1 \cup X_2 \cup \ldots \cup X_k$. If for each $i = 1, 2, \ldots, k$ and for each pair of nonadjacent vertices $x, y \in X_i$, $d(x) + d(y) \ge n$, then G is X-cyclable.*

In 2008, Yamashita generalized Theorem 6 and proved that Theorem 6 is a corollary of the following theorem.

Theorem 7 *([14]). Let G be a 2-connected graph on n vertices and X be a subset of $V(G)$. If $\Delta_2(S) \ge n$ for every independent set S of order $\kappa(X)+1$ in $G[X]$, then G is X-cyclable.*

In 1989, Zhu, Li and Deng [15] found that some vertices may have small degrees, but the vertices around them have large degrees and we may use some large degree vertices to replace small degree vertices in the right position, so that we may construct a longer cycle. This idea leads to the definition of implicit degree.

Definition 1 *([15]).* Let v be a vertex of a graph G and $k = d(v) - 1$. Set $M_2 = \max\{d(u) : u \in N_2(v)\}$. Suppose $d_1 \le d_2 \le d_3 \le \ldots \le d_k \le d_{k+1} \le \ldots$ is the degree sequence of vertices in $N(v) \cup N_2(v)$. If $N_2(v) \ne \emptyset$ and $d(v) \ge 2$, define

$$d^*(v) = \begin{cases} d_{k+1}, & \text{if } d_{k+1} > M_2; \\ d_k, & \text{otherwise,} \end{cases}$$

then the implicit degree of v is defined as $id(v) = \max\{d(v), d^*(v)\}$. If $N_2(v) = \emptyset$ or $d(v) \le 1$, then $id(v) = d(v)$.

Clearly, $id(v) \ge d(v)$ for each vertex v from the definition of implicit degree. Cai [4] gave a polynomial algorithm to calculate implicit degrees of all vertices in a graph. The authors in [15] gave a sufficient condition for a graph to be hamiltonian by using implicit degree sum instead of degree sum in Ore's theorem (Theorem 1).

Theorem 8 *([15]). Let G be a 2-connected graph on $n \ge 3$ vertices. If $id(u) + id(v) \ge n$ for every pair of nonadjacent vertices u and v in G, then G is hamiltonian.*

The introduction of implicit degree is very useful, since many classic results by considering degree conditions for the hamiltonicity of graphs can be generalized to implicit degree conditions, such as [6], [5], etc. In 2011, Li, Ning and Cai used implicit degree sum in place of degree sum in Theorem 6 and got the following result.

Theorem 9 *([10]). Let G be a k-connected graph on n vertices ($k \ge 2$), $X_1, X_2, \ldots, X_k$ be subsets of $V(G)$ and $X = X_1 \cup X_2 \cup \ldots \cup X_k$. If for each $i = 1, 2, \ldots, k$ and for each pair of nonadjacent vertices $x, y \in X_i$, $id(x) + id(y) \ge n$, then G is X-cyclable.*

For a nonempty subset S of $V(G)$, let $i\Delta_k(S) = \max\{\sum_{x \in X} id(x) : X$ is a subset of S with k vertices$\}$. Motivated by the results of Theorem 6 and Theorem 9, we use $i\Delta_2(S)$ in place of $\Delta_2(S)$ in Theorem 7 and obtain the following main result.

Theorem 10. *Let G be a 2-connected graph on n vertices and X be a subset of $V(G)$. If $i\Delta_2(S) \ge n$ for every independent set S of order $\kappa(X)+1$ in $G[X]$, then G is X-cyclable.*

We postpone the proof of Theorem 10 in Section 3. Now we give two remarks, the first one shows that Theorem 9 is a corollary of Theorem 10 and the second one shows that Theorem 10 is stronger than Theorem 7.

Remark 1. Suppose that G is a graph satisfying the condition of Theorem 9, and S is an independent set in X with $\kappa(X)+1$ vertices. Then since $\kappa(X)+1 > k$, there exist two distinct vertices $x, y \in S \cap X_i$ for some i with $1 \le i \le k$. So by the hypothesis of Theorem 9, $id(x) + id(y) \ge n$. Hence G satisfies the condition of Theorem 10.

Remark 2. Let $k \ge 3$ be an odd integer, $G_i = K_{k-1,k}$ be a complete bipartite graph with two partitions X_i and Y_i such that $|Y_i| = |X_i| + 1$ $(i = 1, 2)$ and $E_{\frac{k+1}{2}}$ be a set of $\frac{k+1}{2}$ independent edges. Moreover, $V(G_1)$, $V(G_2)$ and $V(E_{\frac{k+1}{2}})$ are pairwise disjoint. We construct a graph G such that $V(G) = V(G_1) \cup V(G_2) \cup V(E_{\frac{k+1}{2}})$ and $E(G) = E(G_1) \cup E(G_2) \cup E_{\frac{k+1}{2}} \cup \{uv : u \in Y_1 \cup Y_2$ and $v \in V(E_{\frac{k+1}{2}})\} \cup \{xy : x \in Y_1$ and $y \in Y_2\}$. Let $X = X_1 \cup X_2$. Clearly, G is a k-connected graph on $n = 5k - 1$ vertices and $\kappa(X) = k$. By the definition of implicit degree, we can get that $i\Delta_2(S) = 6k > n$ for every independent set S with $\kappa(X) + 1$ vertices in $G[X]$. So by Theorem 10, G is X-cyclable. However, $\Delta_2(S) = 2k < n$ for every independent set S with $\kappa(X)+1$ vertices in $G[X]$. This implies that Theorem 10 is much stronger than Theorem 7.

2. Preliminaries

A path P connecting x and y is called an xy-path and denoted by xPy. For a subgraph H of G, an xy-path P is called an H-path if $V(P) \cap V(H) = \{x, y\}$ and $E(P) \cap E(H) = \emptyset$. A path P is called a *maximal path* of G if the length of each path in G containing P equals the length of P.