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Strong chromatic index of K_4-minor free graphs

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A B S T R A C T

The strong chromatic index $\chi_s'(G)$ of a graph G is the smallest integer k such that G has a proper edge k-colouring with the condition that any two edges at distance at most 2 receive distinct colours. In this paper, we prove that if G is a K_4-minor free graph with maximum degree $\Delta \geq 3$, then $\chi_s'(G) \leq 3\Delta - 2$. The result is best possible in the sense that there exist K_4-minor free graphs G with maximum degree Δ such that $\chi_s'(G) = 3\Delta - 2$ for any given integer $\Delta \geq 3$.

1. Introduction

Only simple graphs are considered in this paper. Let G be a graph with vertex set $V(G)$, edge set $E(G)$, minimum degree $\delta(G)$, and maximum degree $\Delta(G)$. A vertex v is called a *k-vertex* (*k^+-vertex*, *k^--vertex*, respectively) if the degree $d_G(v)$ of v is k (at least k, at most k, respectively). Let $N_G(v)$ and $E_G(v)$ denote the set of vertices adjacent to v and the set of edges incident to v, respectively. It is easy to see that $d_G(v) = |N_G(v)| = |E_G(v)|$ for any vertex v of a simple graph G. If no ambiguity arises in the context, $\delta(G)$, $\Delta(G)$, $d_G(v)$, $N_G(v)$, and $E_G(v)$ are written as $\delta, \Delta, d(v), N(v)$, and $E(v)$, respectively.

A *proper edge k-colouring* of a graph G is a mapping $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges e and e'. The *chromatic index* $\chi'(G)$ of G is the smallest integer k such that G has a proper edge k-colouring. The colouring ϕ is called *strong* if any two edges at distance at most two get distinct colours. Equivalently, each colour class is an induced matching. The *strong chromatic index*, denoted by $\chi_s'(G)$, of G is the smallest integer k such that G has a strong edge k-colouring.

Strong edge colouring of graphs was instructed by Fouquet and Jolivet [11]. It holds trivially that $\chi_s'(G) \geq \chi'(G) \geq \Delta$ for any graph G. In 1985, during a seminar in Prague, Erdős and Nešetřil put forward the following conjecture.

Conjecture 1. *For every graph G with maximum degree Δ,*

$$\chi_s'(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even;} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1), & \text{if } \Delta \text{ is odd.} \end{cases}$$

Erdős and Nešetřil provided a construction showing that Conjecture 1 is tight if it were true. In 1997, using probabilistic method, Molloy and Reed [17] showed that $\chi_s'(G) \leq 1.998\Delta^2$ for a graph G with sufficiently large Δ. The currently best known upper bound for a general graph G is $1.93\Delta^2$, due to Bruhn and Joos [4]. Andersen [1] and independently Horák et al. [13] confirmed Conjecture 1 for the case $\Delta = 3$. If $\Delta = 4$, then Conjecture 1 asserts that

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$\chi_s'(G) \le 20$. However, the best known upper bound is 22 for this case, see [7].

A graph G is called *d-degenerate* if each subgraph of G contains a vertex of degree at most d. Chang and Narayanan [5] proved that if G is a 2-degenerate graph, then $\chi_s'(G) \le 10\Delta - 10$. Recently, Wang [18] strengthened this result to that $\chi_s'(G) \le 6\Delta - 7$ for any 2-degenerate graph G. As a special case, Wang [18] also showed that if all the 3^+-vertices in a graph G induce a forest, then $\chi_s'(G) \le 4\Delta - 3$. Recall that a *chord* in a graph is an edge that joins two nonconsecutive vertices of a cycle. A graph is said to be *chordless* if there is no cycle in the graph that has a chord. Dębski et al. [8] proved that if G is a chordless graph, then $\chi'(G) \le 4\Delta - 3$. More recently, this result was furthermore improved to that $\chi'(G) \le 3\Delta$ for any chordless graph G, see [3].

Suppose that G is a planar graph. Using the Four-Colour Theorem [2] and Vizing Theorem [19], Faudree et al. [10] gave an elegant and short proof to the result that $\chi_s'(G) \le 4\Delta + 4$. They also constructed a class of planar graphs G with $\Delta \ge 2$ and $\chi_s'(G) = 4\Delta - 4$. Let $g(G)$ denote the *girth* of a planar graph G (i.e., the length of a shortest cycle in G). It was shown in [14] that (i) $\chi_s'(G) \le 3\Delta$ if $g(G) \ge 7$; (ii) $\chi_s'(G) \le 3\Delta + 5$ if $g(G) \ge 6$; and (iii) $\chi_s'(G) \le 9$ if $\Delta \le 3$ and $g(G) \ge 6$. The above result (iii) has been recently strengthened to that $\chi_s'(G) \le 9$ for any planar subcubic graph G by Kostochka et al. [15].

In this paper, we focus on studying the strong edge colouring of K_4-minor free graphs. A graph G has a graph H as a *minor* if H can be obtained from a subgraph of G by contracting edges, and G is called *H-minor free* if G does not have H as a minor. A planar graph is called *outerplanar* if it has an embedding in the Euclidean plane such that all the vertices are located on the boundary of the unbounded face. It is shown in [6] that a graph G is an outerplanar graph if and only if G is K_4-minor free and $K_{2,3}$-minor free. Thus, the class of K_4-minor free graphs is a class of planar graphs that contains the class of outerplanar graphs.

Very recently, Hocquard et al. [12] proved that if G is an outerplanar graph with $\Delta \ge 3$ then $\chi_s'(G) \le 3\Delta - 3$ and the upper bound $3\Delta - 3$ is tight. On the other hand, it is easy to see that a K_4-minor free graph G is 2-degenerate by the result of [9] and hence $\chi_s'(G) \le 6\Delta - 7$ by the result of [18]. In this paper, we will show that $\chi_s'(G) \le 3\Delta - 2$ for any K_4-minor free graph G with $\Delta \ge 3$ and the upper bound $3\Delta - 2$ is tight.

2. A structural lemma

Let G be a K_4-minor free graph. Then it was proved in [9] that $\delta \le 2$. For a vertex $u \in V(G)$, let $n_i(u)$ and $n_{i^+}(u)$ denote the number of i-vertices and i^+-vertices that are adjacent to u in G, respectively. A vertex of degree 1 is called a *leaf*. For a vertex $v \in V(G)$, we define

$$D_G(v) = \{y \mid d(y) \ge 3 \text{ such that } vy \in E(G) \text{ or there is a path } vxy \text{ with } d(x) = 2\}.$$

The following key structural lemma appeared in [16]:

Lemma 1. *Let G be a K_4-minor free graph with $\Delta \ge 3$ and $\delta = 2$. Then G contains the following configurations* (A1) *or* (A2).

(A1) *two adjacent 2-vertices.*
(A2) *a vertex v with $d(v) \ge 3$ and $|D_G(v)| \le 2$.*

Lemma 2. *Let G be a K_4-minor free graph with $\Delta \ge 3$. Then G contains one of the following configurations* (B1), (B2) *and* (B3).

(B1) *a vertex v with $n_1(v) \ge 1$ and $n_{2^+}(v) \le 2$.*
(B2) *two adjacent 2-vertices.*
(B3) *a vertex v with $d(v) \ge 3$ and $|D_G(v)| \le 2$.*

Proof. Let G be a connected K_4-minor free graph with $\Delta \ge 3$ that does not contain (B1). Then no 3^--vertex is adjacent to a leaf, and every 3^+-vertex v is adjacent to at most $d(v) - 3$ leaves.

Let H be the graph obtained by removing all leaves from G. Then H is a connected K_4-minor free graph with $1 \le \delta(H) \le 2$. First, assume that $\delta(H) = 1$. Let $v \in V(H)$ with $d_H(v) = 1$. Then $v \in V(G)$ with $d(v) \ge 2$ by the construction of H. Thus, v is adjacent to $d(v) - 1$ (≥ 1) leaves in G, contradicting the previous discussion.

Second, assume that $\delta(H) = 2$. Since (B1) was excluded from G, it follows that $d_H(v) = d(v)$ for any vertex $v \in V(G)$ with $2 \le d(v) \le 3$. Moreover, for each vertex $v \in V(G)$ with $d(v) \ge 4$, we have $n_1(v) \le d(v) - 3$ and so it follows that $d_H(v) = d(v) - n_1(v) \ge 3$. Since $\Delta \ge 3$, it is immediate to derive that $\Delta(H) \ge 3$. By Lemma 1, H contains the configurations (A1) or (A2). If H contains (A1), i.e., two adjacent 2-vertices x and y, then since $d(x) = d_H(x) = 2$, $d(y) = d_H(y) = 2$, and $xy \in E(G)$, both x and y constitute a configuration (B2) in G.

Assume that H contains (A2), i.e., a vertex $v \in V(H)$ with $d_H(v) \ge 3$ and $|D_H(v)| \le 2$. We will show that (i) $d(v) \ge 3$; and (ii) $|D_G(v)| \le 2$, which implies that the vertex v satisfies (B3) in G. Obviously, (i) holds automatically since $d(v) = d_H(v) + n_1(v) \ge d_H(v) \ge 3$. Instead of showing (ii), we only need to prove that $D_G(v) \subseteq D_H(v)$, which implies immediately that $|D_G(v)| \le |D_H(v)| \le 2$.

Let z be an arbitrary vertex in $D_G(v)$. Then $d(z) \ge 3$ by definition, and $d_H(z) \ge 3$ by the previous discussion. If $vz \in E(G)$, then it is easy to see that $vz \in E(H)$, and so $z \in D_H(v)$ by the definition. Thus suppose that $vz \notin E(G)$. So there exists a 2-vertex $w \in V(G)$ such that $zw, wv \in E(G)$. Noting that $d_H(w) = d(w) = 2$, and $zw, wv \in E(H)$, we deduce that $z \in D_H(v)$. In both cases, we get $z \in D_H(v)$. Therefore, $D_G(v) \subseteq D_H(v)$. This completes the proof of the lemma. $\square$

3. Strong edge colouring

This section establishes the main result in this paper. The following two propositions can be affirmed easily.

Proposition 3. *Let P_n be a path with $n \ge 2$ vertices. Then $\chi_s'(P_n) = 1$ if $n = 2$, $\chi_s'(P_n) = 2$ if $n = 3$, and $\chi_s'(P_n) = 3$ if $n \ge 4$.*