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A polynomial-time algorithm for the maximum cardinality cut problem in proper interval graphs

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ABSTRACT

It is known that the maximum cardinality cut problem is NP-hard even in chordal graphs. On the positive side, the problem is known to be polynomial time solvable in some subclasses of proper interval graphs which is in turn a subclass of chordal graphs. In this paper, we consider the time complexity of the problem in proper interval graphs, and propose a polynomial-time dynamic programming algorithm.

1. Introduction

A *cut* of a graph $G = (V(G), E(G))$ is a partition of $V(G)$ into two subsets $S, \bar{S}$ where $\bar{S} = V(G) \setminus S$. The *cut-set* of $(S, \bar{S})$ is the set of edges of G having exactly one endpoint in S. The maximum cardinality cut problem (MaxCut) is to find a cut with a maximum size cut-set, of a given graph.

MaxCut remains NP-hard when restricted to the following graph classes: chordal graphs, undirected path graphs, split graphs, tripartite graphs, co-bipartite graphs [1], unit disk graphs [2] and total graphs [3]. On the positive side, it was shown that MaxCut can be solved in polynomial-time in planar graphs [4], in line graphs [3], in graphs with bounded clique-width [5], and the class of graphs factorable to bounded treewidth graphs [1].

Proper interval graphs are not necessarily planar since any clique (in particular K_5) is a proper interval graph. They are not necessarily line graphs either since the graph $\bar{A}$ consisting of 6 vertices [6] is a proper interval graph, but a forbidden subgraph of line graphs [7]. It is also known that proper interval graphs may have unbounded clique-width [8]. In [9], we have shown that co-bipartite chain graphs are not factorable to bounded treewidth graphs. Since co-bipartite chain graphs are proper interval graphs, this result holds for them too. Therefore, none of the results mentioned in the previous paragraph implies a polynomial-time algorithm for proper interval graphs.

Polynomial-time algorithms for some subclasses of proper interval graphs (also known as indifference graphs) are proposed in [10] and in [9], for split indifference graphs and co-bipartite chain graphs, respectively. A polynomial-time algorithm for proper interval graphs is proposed in [11]. However, as pointed out in [10] (see the paragraph "The Balanced Cut Is not Always Maximal") this algorithm contains a flaw and may return sub-optimal solutions. Consequently, the complexity of MaxCut in proper interval graphs was open. In this work, we generalize the dynamic programming algorithm in [9] to proper interval graphs using the bubble model of proper interval graphs introduced in [12].

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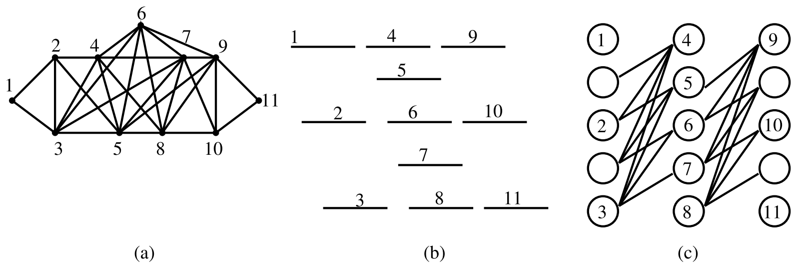

Fig. 1. (a) A proper interval graph G, (b) an interval representation of G, (c) a bubble model of G.

2. Preliminaries

Graph notations and terms: Given a simple graph (no loops or parallel edges) $G = (V(G), E(G))$ and a vertex v of G, uv denotes an edge between two vertices u, v of G. We also denote by uv the fact that $uv \in E(G)$. We denote by $N_G(v)$ the set of neighbors of v in G. Two adjacent vertices u, v of G are *twins* if $N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}$. A vertex having degree zero is termed *isolated*, and a vertex adjacent to all other vertices is termed *universal*. For a graph G and $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of G induced by U, and $G \setminus U \stackrel{def}{=} G[V(G) \setminus U]$. For a singleton $\{x\}$ and a set Y, $Y + x \stackrel{def}{=} Y \cup \{x\}$ and $Y - x \stackrel{def}{=} Y \setminus \{x\}$. A vertex set $U \subseteq V(G)$ is a *clique* (resp. *stable set*) (of G) if every pair of vertices in U is adjacent (resp. non-adjacent). We denote by n be the number of vertices of G.

Some graph classes: A graph is *bipartite* if its vertex set can be partitioned into two stable sets V and V'. We denote such a graph as $B(V, V', E)$ where E is the edge set. A graph G is *co-bipartite* if it is the complement of a bipartite graph, i.e. $V(G)$ can be partitioned into two cliques K, K'. We denote such a graph as $C(K, K', E)$ where E is the set of edges that have exactly one endpoint in K.

A *bipartite chain graph* is a bipartite graph $G = B(V, V', E)$ where V has a nested neighborhood ordering, i.e. its vertices can be ordered as $v_1, v_2, \ldots$ such that $N_G(v_1) \subseteq N_G(v_2) \subseteq \cdots$. V has a nested neighborhood ordering if and only if V' has one [13]. Theorem 2.3 of [14] implies that if $G = B(V, V', E)$ is a bipartite chain graph with no isolated vertices, then the number of distinct degrees in V is equal to the number of distinct degrees in V'.

A co-bipartite graph $G = C(K, K', E)$ is a *co-bipartite chain* (also known as co-chain) graph if K has a nested neighborhood ordering [15]. Since $K \subseteq N_G(v)$ for every $v \in K$, the result for chain graphs implies that K has a nested neighborhood ordering if and only if K' has such an ordering.

A graph G is *interval* if its vertices can be represented by intervals on a straight line such that two vertices are adjacent in G if and only if the corresponding intervals are intersecting. An interval graph is *proper* (resp. *unit*) if it has an interval representation such that no interval properly contains another (resp. every interval has unit length). It is known that the class of proper interval graphs is equivalent to the class of unit interval graphs [16].

Cuts: We denote a cut of a graph G by one of the subsets of the partition. $E(S, \bar{S})$ denotes the *cut-set* of S, i.e. the set of the edges of G with exactly one endpoint in S, and $cs(S) \stackrel{def}{=} \left|E(S, \bar{S})\right|$ is termed the *cut size* of S. A maximum cut of G is one having the biggest cut size among all cuts of G. We refer to this size as the *maximum cut size* of G. Clearly, S and $\bar{S}$ are dual; we thus can replace S by $\bar{S}$ and $\bar{S}$ by S everywhere. In particular, $E(S, \bar{S}) = E(\bar{S}, S)$, and $cs(S) = cs(\bar{S})$.

Bubble models: A *2-dimensional bubbles structure* $\mathcal{B}$ for a finite non-empty set A is a 2-dimensional arrangement of bubbles $\{B_{i,j} \mid j \in [k], i \in [r_j]\}$ for some positive integers $k, r_1, \ldots r_k$, such that $\mathcal{B}$ is a near-partition of A. That is, $A = \cup\mathcal{B}$ and the sets $B_{i,j}$ are pairwise disjoint, allowing for the possibility of $B_{i,j} = \emptyset$ for arbitrarily many pairs i, j. For an element $a \in A$ we denote by $i(a)$ and $j(a)$ the unique indices such that $a \in B_{i(a),j(a)}$.

Given a bubble structure $\mathcal{B}$, the graph $G(\mathcal{B})$ defined by $\mathcal{B}$ is the following graph:

i) $V(G(\mathcal{B})) = \cup\mathcal{B}$, and
ii) $uv \in E(G(\mathcal{B}))$ if and only if one of the following holds:
   - $j(u) = j(v)$,
   - $j(u) = j(v) + 1$ and $i(u) < i(v)$.

$\mathcal{B}$ is a *bubble model* for $G(\mathcal{B})$.

A *compact representation* for a bubble model is an array of *columns* each of which contains a list of non-empty bubbles, and each bubble contains its row number in addition to the vertices in this bubble.

Theorem 2.1. *[12]*

i) *A graph is a proper interval graph if and only if it has a bubble model.*
ii) *A bubble model for a graph on n vertices contains $O(n^2)$ bubbles and it can be computed in $O(n^2)$ time.*
iii) *A compact representation of a bubble model for a graph on n vertices can be computed in $O(n)$ time.*

Note that the set of vertices in two consecutive columns in $\mathcal{B}$ induces a co-bipartite chain graph. In other words, a proper interval graph can be seen as a chain of co-bipartite chain graphs, see Fig. 1. Using this observation, we generalize our result in [9]. To keep the analysis simpler, we use the standard representation of the bubble model, since using the compact representation does not improve the overall running time of the algorithm.