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Some notes on bounded starwidth graphs

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ABSTRACT

We introduce the graph parameter starwidth. We show results on characterization, complexity and the relation with other parameters. We also discuss the complexity of problems on bounded starwidth graphs.

1. Introduction

In this paper, we introduce the graph parameter starwidth. It naturally measures how much the graph looks like a star in the sense that it is the star-equivalent of treewidth [2]. A star decomposition of a graph and its starwidth are defined as follows.

Definition 1. A star decomposition of a graph $G = (V, E)$ is a star with core X_0 and leafs $X_1, \dots, X_m$, where each X_i is associated with a subset of V, satisfying the following properties:

1. The union of all sets X_i equals V.
2. For every edge (v, w) in G, there is a subset X_i that contains both v and w.
3. For $i \neq j, i, j \geq 1$, if X_i and X_j both contain vertex v, then X_0 contains v as well.

The width of a star decomposition is defined as $\max_i |X_i| - 1$. The starwidth $sw(G)$ of graph G is the minimum width among all possible star decompositions of G. An example of a star decomposition of a graph is illustrated in Fig. 1.

Note that star graphs have starwidth equal to 1. Also note that the starwidth of a graph is greater than or equal to its treewidth, since every star decomposition is also a tree decomposition. It is interesting to study this parameter, because it might be that certain problems which are intractable on graphs restricted to some parameter being bounded, like bounded treewidth graphs, become tractable on bounded starwidth graphs. Unfortunately, we did not succeed in finding such a problem. However, this does not mean that such a problem does not exist and we think that the results obtained in this work can be interesting in their own right.

Next, we start with characterizing the class of graphs of starwidth equal to 1 with a set of forbidden minors. We also show that the obstruction set for bounded starwidth graphs contains at least a path and a galaxy (forest of stars). Then, we show that computing the starwidth of a given graph is a NP-hard problem. It is fixed parameter tractable when parametrized by its outcome. We also show how the parameter relates to tree-depth and vertex cover number. Finally, we discuss the complexity of some problems on bounded starwidth graphs.

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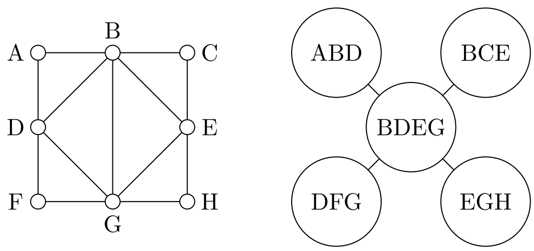

Fig. 1. Example graph and a star decomposition with width 3.

2. Characterization

We start with stating that the class of bounded starwidth graphs is closed under taking minors. It is easy to see that this is true, since deletion of vertices and edges or contraction of edges does not increase the starwidth of a graph.

Lemma 1. *If H is a minor of G, $sw(H) \le sw(G)$.*

This enables us to use the Graph Minor Theorem [11]. This states that every class of minor-closed graphs can be characterized by a finite set of forbidden minors, also known as the obstruction set. We now show that the obstruction set of the graphs of starwidth 1 is given by the graphs in Fig. 2.

Theorem 1. *Graph G has $sw(G) = 1$ iff G does not have K_3,P_5,$P_4 + P_3$ and $3P_3$ as minor.*

Proof. It is easy to see that K_3, P_5, $P_4 + P_3$ and $3P_3$ have starwidth 2. Since the class of graphs of starwidth equal to 1 is closed under taking minors, these graphs are forbidden minors.

Graphs that do not have K_3 as a minor are forests. Moreover, graphs that do not have P_5 as a minor have a diameter of at most 3 in each component. If there is a component with diameter 3, all other components are edges or vertices. Otherwise, $P_4 + P_3$ is a minor. Similarly, if there is no component with diameter 3, the graph contains at most two stars with more than two vertices, and the other components are edges or vertices. Otherwise, $3P_3$ is a minor. The theorem follows because the graphs described, i.e. graphs where at most two vertices have degree at least 2, have starwidth equal to 1. $\square$

We now discuss the obstruction set for graphs of starwidth p, for some fixed p. Before proceeding to the next theorem, we define the graph class *galaxy*. A graph is called a galaxy if each component is a star. In particular, we define the (n, k)-galaxy to be a graph having n components, where each component is a star on k vertices.

Theorem 2. *The obstruction set characterizing graphs of starwidth p contains at least one path and one galaxy.*

Proof. First, we show that a path graph with n vertices has starwidth $\Omega(\sqrt{n})$. The optimal way to construct a star decomposition for P_n is to pick k vertices that split the path into $O(n/k)$ approximate equally sized segments, and put them in the core. This decomposition has starwidth $O(\min\{k, n/k\})$, which is minimized for $k = O(\sqrt{n})$. Hence, we have that $sw(P_n) \in \Omega(\sqrt{n})$. Thus, in order to get bounded starwidth, we need at least one path in the obstruction set.

Secondly, it is easy to see that the (n, n)-galaxy has starwidth exactly $n - 1$. This can be obtained by taking all centers in the core. Leaving one out leads to having a leaf containing n vertices. Hence, we need at least one galaxy in the obstruction set. $\square$

It remains open whether these two forbidden minors define the class of bounded starwidth graphs, i.e. whether the class of bounded starwidth graphs is equivalent to the minor-closed class of graphs with an obstruction set containing at least one path and one galaxy.

3. Complexity

In this section, we first show that computing the starwidth of a given graph is NP-hard. Secondly, we will show that the problem parametrized by its outcome is fixed parameter tractable and even solvable in linear time.

To show that computing starwidth is NP-hard, we reduce from the Vertex Cover problem [6]. Here, we are given a graph $G = (V, E)$ and an integer k. The question is whether there is a subset of the vertices of size at most k such that every edge is covered, i.e. it has at least one endpoint in this set. The problem is still NP-complete in some restricted setting.

Lemma 2. *The Vertex Cover problem is still NP-complete when each vertex has degree less than k, and there are no two vertices u, v with degree $k - 1$ such that $(u, v) \in E$ and $N(u) \setminus \{v\} = N(v) \setminus \{u\}$, where $N(u)$ denotes the neighborhood of vertex u.*

Proof. First of all, the problem is still NP-complete when each vertex has degree less than k. If vertex v has degree at least k, it is easy to verify whether the neighbors of v form a vertex cover. If this turns out to be false, we include v in the vertex cover, remove it from G and lower k by 1. Secondly, the problem is NP-complete when there are no two vertices of degree $k-1$ connected with an edge sharing the same $k - 2$ neighbors. If an instance does have such a pair, one can do the following preprocessing step. If exactly one of the vertices is included in the vertex cover, then all the other $k - 2$ neighbors should be included in

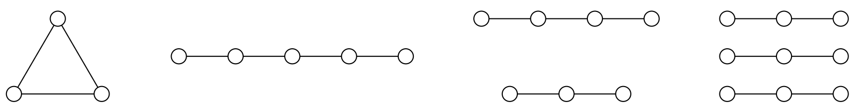

Fig. 2. K_3, P_5, $P_4 + P_3$ and $3P_3$.