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# Strong Menger connectivity with conditional faults of folded hypercubes

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ABSTRACT

Motivated by parallel routing in networks with faults and evaluating the reliability of networks, we consider strong Menger connectivity of the folded hypercube networks. We show that in all $n$-dimensional folded hypercubes with a vertex set $S$ of $n-1$ vertices removed, each pair of unremoved vertices $x$ and $y$ are connected by $\min\{d_{G-S}(x), d_{G-S}(y)\}$ vertex-disjoint paths (i.e., strong Menger property), where $d_{G-S}(x)$ and $d_{G-S}(y)$ are the remaining degree of vertices $x$ and $y$ in $G-S$, respectively. Moreover, if there are $2n-3$ vertex faults, and each vertex except for the vertex faults has at least two fault-free adjacent vertices, then all folded hypercube networks still have the strong Menger property.


## 1. Introduction

The connectivity is one of the important parameters to evaluate the reliability and fault tolerance of a network [2,3,14]. The traditional connectivity $\kappa(G)$ of a graph $G$ is defined as the minimum number of vertices whose removal from $G$ induces a disconnected graph. In contrast to the concept, Menger's Theorem [10] provided a local point of view, and define the connectivity of any two vertices as the minimum number of internally vertex-disjoint paths between them. As an interconnection network is usually modeled by a connected graph in which vertices represent processors and edges represent links between processors. The connectivity is useful to assess the reliability of the interconnection networks.

As the vertex-disjoint paths can not only increase the efficiency of message transmission, but also it can provide alternative paths when there are some vertex faults. So vertex-disjoint path systems play an important role on the parallel routing in networks. Menger's Theorem [10] is a classic result in connectivity and states that if a graph $G$ is $k$-connected, then every pair of vertices in $G$ is connected by $k$ vertex-disjoint paths.

With the increasing size of networks, the vertex faults have become common. Suppose that the network $G$ has a set $S$ of faulty vertices. Let $u$ and $v$ be two non-faulty vertices in $G$, and we know the numbers of the non-faulty neighbours of $u$ and $v$ are $d_{G-S}(u), d_{G-S}(v)$, respectively. We are interested in constructing the maximum number of vertex-disjoint fault-free paths between $u$ and $v$. Obviously, the vertex-disjoint paths between $u$ and $v$ can not exceed $min\{d_{G-S}(u), d_{G-S}(v)\}$, where $d_{G-S}(u)$ and $d_{G-S}(v)$ are the remaining degree of vertices $u$ and $v$ in $G-S$, respectively. Note that the problem has a strong similarity to that of Menger's theorem. Thus it was introduced the definition of strong Menger-connectivity by Oh and Chen [6].

We study strong Menger-connectivity of regular graphs. The $k$-regular graph is *strongly Menger-connected* if for any copy $G-F$ of $G$ with at most $k-2$ vertices removed, each pair $u$ and $v$ of $G-F$ are connected by $\min\{d_{G-F}(u), d_{G-F}(v)\}$ vertex-disjoint paths, where

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$d_{G-F}(u)$ and $d_{G-F}(v)$ are the remaining degree of vertices $u$ and $v$, respectively. This concept is firstly applied on hypercubes and introduced by Oh and Chen [6]. And later Shih et al. showed that the hypercube-like networks are strongly Menger-connected [12]. A $k$-regular graph is maximally connected if it is $k$-connected. It is not hard to see that the strong Menger-connectivity is a generalization of the maximal connectivity. It is easy to construct a maximally connected graph that is not strongly Menger-connected.

In this paper, we study the strong Menger-connectivity of folded hypercubes. Moreover, if there are $2n-3$ vertex faults, and each vertex except for the vertex faults has at least two fault-free adjacent vertices, then all folded hypercube networks still have the strong Menger property.

## 2. Preliminary

The hypercube is one of the most famous interconnection network models. An $n$-dimensional hypercube is an undirected graph $Q_n = (V, E)$ with $|V| = 2^n$ and $|E| = n2^{n-1}$. Each vertex can be represented by an $n$-bit binary string. There is an edge between two vertices whenever their binary string representation differs in only one bit position. Following Latifi et al. [8], we express $Q_n$ as $D_0 \bigodot D_1$, where $D_0$ and $D_1$ are two $n-1$ cubes of $Q_n$ induced by the vertices with the ith coordinates 0 and 1 respectively. Sometimes we use $X_1 \cdots X_{i-1}0X_{i+1} \cdots X_n$ to denote a $(n-1)$-dimensional subcube of $Q_n$, where $X \in Z_2$. Moreover, if $n-k$ coordinates of the $n$-bit binary string are fixed, we can use $X_1X_2 \cdots X_k$ to denote a $k$-dimensional subcube in $Q_n$. Clearly, the vertex $v$ in one $(n-1)$-subcube has exactly one neighbour $v'$ in the other $(n-1)$-subcube. Let $u = x_1x_2 \cdots x_i \cdots x_n \in V(Q_n)$, we use $\bar{u} = \bar{x_1}\bar{x_2} \cdots \bar{x_n}$ to denote a vertex that all coordinates of $\bar{u}$ are different from $u'$s, i.e., $\bar{x_i}$ represents the complement of $x_i$.

As one of the important variants of the hypercube network, the $n$-dimensional folded hypercube $FQ_n$, proposed by El-Amawy [5], is obtained from an $n$-dimensional hypercube $Q_n$ by adding an edge between any pair of vertices with complementary addresses. There are some results about the folded hypercubes, one can refer to [4,15,9,11,16–18] for the detail. The folded hypercube $FQ_n$ is superior to $Q_n$ in some properties, see [5, 7]. Thus the folded hypercube is an enhancement on the hypercube $Q_n$. $FQ_n$ is obtained by adding a perfect matching $M$ on the hypercube, where $M = \{(u, \overline{u}) | u \in V(Q_n)\}$. In addition, denote by $M_i$ the edge set $\{(x_1x_2 \cdots x_{i-1}x_ix_{i+1} \cdots x_n, x_1x_2 \cdots x_{i-1}\overline{x_i}x_{i+1} \cdots x_n) | \overline{x_i}$ represents the complement of $x_i\}$. One can see that $E(Q_n) = \bigcup_{i=1}^{n} M_i$ and $E(FQ_n) = E(Q_n) \bigcup M$. The 3-dimensional and 4-dimensional folded hypercubes are shown in the following Fig. 1 and Fig. 2, respectively.

For convenience, $FQ_n$ can be expressed as $D_0 \bigotimes D_1$, where $D_0$ and $D_1$ are $(n-1)$-dimensional subcubes induced by the vertices with the $i$th coordinate 0 and 1 respectively. Let $v$ be a vertex of a graph $G$, we use $N_G(v)$ to denote the vertices that are adjacent to $v$. Let $u, v \in V(G)$, $d(u, v)$ denote the distance between $u$ and $v$. Let $A \subseteq V(G)$, we denote by $N_G(A)$ the vertex set $\bigcup_{v \in V(A)} N_G(v) \setminus V(A)$ and $C_G(A) = N_G(A) \bigcup A$. And also, we use $\theta_G(g)$ to denote the minimum number of vertices that are adjacent to a vertex set of $g$ vertices in $G$. Let $g(G)$ denote the girth of the graph $G$, that is, the shortest cycle in $G$. A graph $G$ is $k$-regular if every vertex in $G$ has degree $k$. A graph $G$ is connected if any two vertices of $V(G)$ contain at least one path. The vertex set $S$ of $V(G)$ is a cut set if $G - S$ is disconnected. The connectivity of $G$, denoted by $\kappa(G)$, is defined as the minimum size of a vertex cut if $G$ is not a complete graph, and $\kappa(G) = |V(G)| - 1$ if otherwise. We say that a graph $G$ is $k$-connected if $\kappa(G) \geq k$. A graph has connectivity $k$ if the graph $G$ is $k$-connected but not $(k+1)$-connected. For the terminologies, one can refer to [1] for the detail.

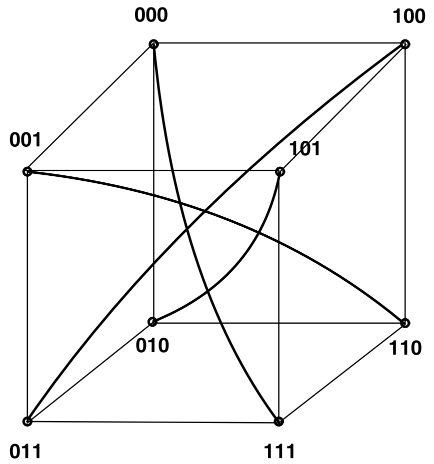


**Fig. 1.** The 3-dimensional Folded hypercube.

It is known that the connectivity of $FQ_n$ is $n+1$. So we study the strongly Menger connectivity of $FQ_n$ with at most $n-1$ vertices deleted. Moreover, if we restrict a condition such that each vertex has at least two fault-free adjacent vertices, all folded hypercubes are still strong Menger-connected, even if there are up to $2n-3$ vertex faults.

## 3. Strong Menger connectivity of folded hypercubes

In this section, we will show that the folded hypercubes are strongly Menger-connected if there are at most $n-1$ vertex faults. Before proving this result, we need the following lemmas.

**Lemma 3.1** *([13,20]). Let $S$ be a vertex set in $V(Q_n)$ with $|S| = g$, then*

$\theta_{Q_n}(g) = -\frac{1}{2}g^2 + (n - \frac{1}{2})g + 1$ *for* $1 \leq g \leq n+1$,

$\theta_{Q_n}(g) = -\frac{1}{2}g^2 + (2n - \frac{3}{2})g - n^2 + 2$ *for* $n+2 \leq g \leq 2n$.

**Lemma 3.2** *([19]). Let $n \geq 4$ and $F \subseteq V(Q_n)$. Then the following holds.*

*(i) If $|F| < \theta_{Q_n}(g)$ and $1 \leq g \leq n-3$, then $Q_n - F$ contains exactly one large component of order at least $2^n - |F| - (g-1)$.*

*(ii) If $|F| < \theta_{Q_n}(g)$ and $n-2 \leq g \leq n+1$, then $Q_n - F$ contains exactly one large component of order at least $2^n - |F| - (n+1)$.*