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3-colouring for dually chordal graphs and generalisations

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ABSTRACT

In this paper, we investigate the Colourability problem for dually chordal graphs and some of its generalisations. We show that the problem remains NP-complete if limited to four colours. For the case of three colours, we present a simple linear time algorithm for dismantlable graphs (which include dually chordal graphs) and present an $\mathcal{O}(n^3m)$ time algorithm for graphs with tree-breadth 1. Additionally, we show that a dually chordal graph is 3-colourable if and only if it is perfect and has no clique of size four.

1. Introduction

Colouring a graph is the problem of assigning a minimum number of colours to the vertices of a given graph such that adjacent vertices have distinct colours. It is a classic problem in computer science and one of Karp's 21 NP-complete problems [8]. Colouring remains NP-complete if limited to three colours [10]. However, it is well known that 2-Colourability can be solved in linear time for any graph.

Dually chordal graphs, introduced by Brandstädt et al. [4], are closely related to chordal graphs, hypertrees and acyclic hypergraphs, which also explains the name of this graph class. As we will observe in Section 3, Colouring remains NP-complete for dually chordal graphs, even if limited to four colours.

A generalisation of dually chordal graphs, introduced by Dragan and Köhler in [6], are graphs with tree-breadth 1. Informally, a graph G has tree-breadth 1 if its vertices can be decomposed into bags such that these bags form a tree, each vertex and each edge of G is contained in a bag, for each vertex, the bags containing it induce a subtree, and each bag is the subset of the neighbourhood of some vertex in G. We will give a formal definition is Section 4.

In Section 3 of this paper, we will show that the 3-Colouring problem is solvable in linear time for dually chordal graphs and some generalisations. Then, in Section 4, we will show that the problem solvable in polynomial time for graphs with tree-breadth 1.

2. Preliminaries

In this paper, all graphs are finite, undirected, connected, and without loops or multiple edges. Let $G = (V, E)$ be a graph with the vertex set V and the edge set E. We use $n = |V|$ and $m = |E|$ to denote the cardinality of the vertex set and edge set of G. For a set $U \subseteq V$, $G[U]$ denotes the induced subgraph of G with the vertex set U.

A set of vertices S is a *clique* if all distinct vertices $u, v \in S$ are pairwise adjacent. A clique of size i is denoted as K_i. The number of vertices in the largest clique of G is the *clique number* of G. An *independent set* is a vertex set I such that all vertices $u, v \in I$ are pairwise non-adjacent. A *chordless cycle* C_k is a subgraph containing k vertices $\{v_1, \ldots, v_k\}$ such that two vertices v_i and v_j are adjacent if and only if $|i - j| = 1$ (index arithmetic

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modulo k). Accordingly, in the complement $\overline{C_k}$ of a chordless cycle, v_i and v_j are adjacent if and only if $|i - j| > 1$.

Let $N_G(v) = \{u \in V \mid uv \in E\}$ denote the *open* and $N_G[v] = N_G(v) \cup \{v\}$ the *closed neighbourhood* of the vertex v in the graph G. If no ambiguity arises, we will omit the subscript G. A vertex u is *universal* in G if $N[u] = V$.

A graph is *locally connected* if, for all vertices v, the open neighbourhood $N(v)$ induces a connected graph; in this case, we say $N(v)$ is connected. A vertex set S is a *separator* for G, if $G[V \setminus S]$ is no longer connected. A vertex v is an *articulation point* of G if $\{v\}$ is a separator. Maximal subgraphs without articulation points are called *blocks*.

A graph G is *k-colourable* if V can be partitioned into k independent sets. The *chromatic number* of a graph G is the lowest k such that G is k-colourable. The *k-Colourability* problem asks if a given graph is k-colourable. A graph G is *perfect* if the chromatic and the clique number are equal for each induced subgraph H of G.

3. Dually chordal graphs

Definition 1 *([4]).* A graph G is *dually chordal* if it admits a spanning tree T such that, for each vertex v in G, $N[v]$ induces a subtree in T.

We will call the tree T defined in Definition 1 the *characteristic spanning tree* of T. It is easy to see that an arbitrary graph becomes dually chordal if a universal vertex u is added: Make every vertex $v \neq u$ a leaf of T which is adjacent to u. This leads to a reduction for several NP-complete problems, including Colourability. The idea for this method was already given by Brandstädt et al. [3].

Observation 1. *It is NP-complete to decide if a dually chordal graph is 4-colourable.*

Proof. Let $G = (V, E)$ be an arbitrary graph. Thus, $G' = (V \cup \{u\}, E \cup \{uv \mid v \in V\})$ with $u \notin V$ is dually chordal. Because u cannot share a colour with any other vertex, G is 3-colourable if and only if G' is 4-colourable. Therefore, it is NP-complete to decide if a dually chordal graph is 4-colourable. □

Clearly, a graph has to be K_4-free to be 3-colourable. The next theorem investigates colourabilty of K_4-free dually chordal graphs.

Theorem 2. *A dually chordal graph G is 3-colourable if and only if G is perfect and K_4-free.*

Proof. By definition, a perfect graph is 3-colourable if and only if it is K_4-free. Additionally, the Strong Perfect Graph Theorem [5] states that a graph is perfect if and only if it does not contain a C_k or $\overline{C_k}$ for any odd $k \geq 5$. Hence, showing that a given 3-colourable dually chordal graph does not contain a C_k or $\overline{C_k}$ for any odd $k \geq 5$ is sufficient to prove Theorem 2.

Let G be a 3-colourable dually chordal graph with a characteristic spanning tree T. For two vertices u and v in G, let $P_T[u, v]$ denote the shortest path from u to v in T.

Claim. *If u and v are adjacent in G, then $P_T[u, v]$ induces a clique.*

Proof (Claim). Let $P_T[u, v] = \{u = t_1, t_2, \dots, t_k = v\}$ such that t_i is adjacent to t_{i+1} in T. Note that, by Definition 1, $P_T[x, y] \subseteq N[x] \cap N[y]$ for all adjacent vertices x and y in G. Therefore, for every vertex $t_i \in P_T[u, v]$, $P_T[u, v] = P_T[u, t_i] \cup P_T[t_i, v] \subseteq N[t_i]$. ♢

For $k \geq 7$, no $\overline{C_k}$ is 3-colourable. Hence, G does not contain any $\overline{C_k}$ for $k \geq 7$. Assume that, for some odd $k \geq 5$, G contains a chordless cycle $\mathcal{C} = \{c_1, \dots, c_k\}$ with the edges $c_i c_{i+1}$ (index arithmetic modulo k). Recall that $C_5 = \overline{C_5}$.

Claim. *There is a vertex w in G with $\mathcal{C} \subseteq N(w)$.*

Proof (Claim). Because G is 3-colourable, it does not contain a K_4. Hence, for every edge $c_i c_{i+1}$, $P_T[c_i, c_{i+1}]$ is a clique with cardinality at most 3. Let $P_T[c_i, c_{i+1}] = \{c_i, w_i, c_{i+1}\}$ with $c_i = w_i$ if and only if $c_i c_{i+1}$ is an edge in T. Thus, for all consecutive vertices u and v in the sequence $\sigma = \langle c_1, w_1, c_2, \dots, w_k, c_1 \rangle$, either $u = v$ or uv is an edge in T. Let T_σ be the subtree of T induced by σ.

Since $\mathcal{C}$ is chordless and $k \geq 5$, $c_i \neq w_j$ if $|i - j| \geq 2$. Otherwise, c_i and c_j are adjacent. It follows that σ describes a traversal of T_σ in which every vertex c_i is visited at most once, even if $c_i = w_i$. Thus, each c_i is a leaf of T_σ. Because $k \geq 5$, no two leaves of T_σ are adjacent. Hence, for all i, $c_i \neq w_i$ and, since c_i is adjacent to w_{i-1} and w_i, $w_{i-1} = w_i$. Therefore, there is a vertex $w = w_1 = \ldots = w_k$ in G with $\mathcal{C} \subseteq N(w)$. ♢

Sine $\mathcal{C}$ is and odd cycle, it requires at least three colours for every valid colouring of G. However, because there is a vertex w with $\mathcal{C} \subseteq N(w)$, a fourth colour would be required for w. Therefore, G does not contain any odd cycles of length $k \geq 5$. This completes the proof. □

Next, we will show that it is possible to find a 3-colouring for dually chordal graphs and some generalisations in linear time. To do so, we will show first that, for these graphs, each block is locally connected. Then, Algorithm 1 computes a 3-colouring in linear time for graphs whose blocks are locally connected.

For a given graph $G = (V, E)$, let $\sigma = \langle v_1, v_2, \dots, v_n \rangle$ be a vertex ordering for G with $V_i = \{v_i, v_{i+1}, \dots, v_n\}$, $N_i(v) = N(v) \cap V_i$, and $N_i[v] = N[v] \cap V_i$. Then, σ is called *dismantling ordering* for G if, for all $v_i \neq v_n$, there is a vertex $v_j \in V_{i+1}$ with $N_i[v_i] \subseteq N_i[v_j]$. G is called *dismantlable* if it admits a dismantling ordering σ. Dismantlable graphs are also called *cop-win* graphs. Subclasses are for example dually chordal graphs [4], Helly graphs [2], and bridged graphs [1].

Lemma 3. *Blocks of dismantlable graphs are locally connected.*

Proof. Assume that we are given a dismantlable graph $G = (V, E)$ and a dismantling ordering $\sigma = \langle v_1, v_2, \dots, v_n \rangle$ for G. Additionally, assume by induction that all blocks of $G[V_{i+1}]$ are locally connected.