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Directed hypergraphs and Horn minimization

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ABSTRACT

A Boolean function given in a conjunctive normal form is Horn if every clause contains at most one positive literal, and it is pure Horn if every clause contains exactly one positive literal. Due to their computational tractability, Horn functions are studied extensively in many areas of computer science and mathematics such as combinatorics, artificial intelligence, database theory, algebra and logic.

The present paper considers the problem of finding minimal representations of pure Horn functions. We give a new proof for a recent min–max result of Boros et al. regarding body-minimal representations. The proof is algorithmic and finds the so called Guigues–Duquenne basis. We also describe a new construction that combines two existing representations into a third one.

1. Introduction

As a subclass of Boolean functions, Horn functions play an important role in different areas of mathematics due to their interesting computational properties. The satisfiability problem for this subclass of Boolean functions can be solved in linear time and the equivalence of Horn formulas can be decided in polynomial time [12]. This concept appears as lattices and closure systems in algebra, as implicational systems in artificial intelligence, as directed hypergraphs in graph theory, and is also used for representing knowledge base in propositional expert systems.

Informally, the Horn minimization problem is to find a minimal representation that is equivalent to a given Horn formula. For example, such a representation can be used to reduce the size of the knowledge base in a propositional expert system, thus improving the performance of the system. The size of a formula can be measured in many different ways (see [5]). Unfortunately, it is NP-hard to find an optimal representation for almost all of these measures. There is however an interesting exception, called body-minimal representation, for which polynomial time algorithms were independently discovered [5,11,13]. In [7], Boros et al. gave an explanation why this measure is so different from the others in terms of tractability by providing a min–max result on the minimum number of bodies appearing in the representation of a Horn function. Their proof is algorithmic and it actually determines a canonical body-minimal representation called the Guigues–Duquenne basis.

A common aspect of previous algorithms for determining a body-minimal representation is that they are using frameworks different from that of directed hypergraphs, for example, functional dependencies or implication systems. For this reason, the steps of these algorithms are difficult to follow and they do not reveal the structure of body-minimal representations. Our aim was to give a better understanding of the min–max result of [7] by using a purely graph theoretical approach.

In contrast to body-minimal representations, edge-minimal representations are not only hard to find but even hard to approximate. Bhattacharya et al. [6] showed that this problem is inapproximable within a factor $2^{O(\log^{(1-\varepsilon)}(n))}$ assuming $NP \subsetneq DTIME(n^{polylog(n)})$, while Boros and Gru-

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ber showed that it is inapproximable within a factor $2^{O(\log^{1-o(1)} n)}$ assuming $P \subsetneq NP$, where n denotes the number of variables. However, the existence of an $O(n^c)$ approximation for some $0 < c < 1$ is a rather interesting open problem; such an approximation algorithm would immediately find a wide list of applications. We present a surprising result, which given two pure Horn formulas Φ_1 and Φ_2, constructs a new one Φ such that the bodies and heads of Φ form subsets of the bodies of Φ_1 and the heads of Φ_2, respectively. We hope that this observation may help us in finding a good approximation for the edge-minimal representation.

The rest of the paper is organized as follows. A brief introduction into Horn logic is given in Section 2. We give a new algorithmic proof of the min–max result of Boros et al. in Section 3. In Section 4, we show that the body-minimal representation provided by the algorithm is in fact the GD basis. Finally, we show how a new representation from two given ones can be constructed in Section 5.

2. Preliminaries

2.1. Horn logic

Let V be a set of n variables. Members of V are called **positive** while their negations are called **negative literals**. A **Boolean function** is a mapping $f : \{0,1\}^V \to \{0,1\}$. For a subset $Z \subseteq V$ let χ_Z denote the **characteristic vector** of Z, that is, $\chi_Z(v) = 1$ if $v \in Z$ and 0 otherwise. Then Z is called **true** for f if $f(\chi_Z) = 1$ and **false** otherwise. The **sets of true** and **false sets** of f are denoted by $\mathcal{T}_f$ and $\mathcal{F}_f$, respectively.

It is known that any Boolean function can be represented by a **conjunctive normal form** (CNF). A CNF is a conjunction of **clauses**, where a clause is a disjunction of literals. A clause is **Horn** if at most one of its literals is positive, and is **pure Horn** (or **definite Horn**) if it contains exactly one positive literal. We usually denote the **set of clauses** appearing in a representation by $\mathcal{C}$. A CNF $\Phi = (V, \mathcal{C})$ is **pure Horn** if all of its clauses are pure Horn. Finally, a Boolean function h is **pure Horn** if it can be represented by a pure Horn CNF. For a subset $\emptyset \neq B \subseteq V$ and $v \in V \setminus B$ we write $(B \to v)$ to denote the pure Horn clause $C = v \vee \bigvee_{u \in B} \overline{u}$. Here B and v are called the **body** and **head** of the clause, respectively. The **set of bodies** and **set of heads** appearing in a CNF representation Φ are denoted by $\mathcal{B}(\Phi)$ and $\mathcal{H}(\Phi)$, respectively.

It is known that for any pure Horn function h, $\mathcal{T}_h$ is closed under intersection and contains V (see e.g. [9]). Vice versa, for any set $\mathcal{T}$ of subsets of V which is closed under intersection and contains V, there exists a pure Horn function h with $\mathcal{T}_h = \mathcal{T}$. Indeed, the CNF $\Phi = (V, \mathcal{C})$ with $\mathcal{C} = \{(B \to v) : \nexists T \in \mathcal{T} \text{ s.t. } B \subseteq T,\ v \notin T\}$ represents such a pure Horn function h. As any Boolean function is uniquely determined by its set of true sets, the above implies that there is a one-to-one correspondence between pure Horn functions and sets of subsets of V closed under intersection and containing V.

Given a pure Horn function h, the **forward chaining closure** of a set $Z \subseteq V$ is the unique smallest true set containing Z and is denoted by $F_h(Z)$. If Φ is a pure Horn CNF representation of h then the forward chaining closure can be obtained by the following method. Set $F_\Phi^0(Z) := Z$. In a general step, if $F_\Phi^i(Z)$ is a true set then $F_h(Z) = F_\Phi^i(Z)$. Otherwise take an arbitrary violated implication $(B \to v)$ of Φ and set $F_\Phi^{i+1} := F_\Phi^i(Z) + v$. Note that $(B \to v)$ is violated by $F_\Phi^i(Z)$ if and only if $B \subseteq F_\Phi^i(Z)$ but $v \notin F_\Phi^i(Z)$. The result of the process depends neither on the particular choice of the representation Φ nor on the order in which violated implications are chosen, but only on the underlying function h.

2.2. Directed hypergraphs

Directed hypergraphs are generalizations of directed graphs and can be defined in several ways [10,14]. In our investigations we will use the following notation. A **directed hypergraph** is a pair $H = (V, \mathcal{E})$ where V is a set of **nodes** and $\mathcal{E}$ is a set of **hyperedges**. A hyperedge is a pair (B, v) where $\emptyset \neq B \subseteq V$ is the **body** and $v \in V \setminus B$ is the **head** of the hyperedge. The **set of bodies** and **set of heads** appearing in H are denoted by $\mathcal{B}(H)$ and $\mathcal{H}(H)$, respectively. We say that a hyperedge $(B, v) \in \mathcal{E}$ **covers** a set $Z \subseteq V$ if $B \subseteq Z$ and $v \notin Z$. The hypergraph H **covers** a family $\mathcal{P}$ of subsets of V if for each $Z \in \mathcal{P}$ there exists an edge in $\mathcal{E}$ covering Z. A subset $Z \subseteq V$ is called **true** if H does not cover Z and **false** otherwise. The **sets of true** and **false sets** are denoted by $\mathcal{T}_H$ and $\mathcal{F}_H$, respectively.

Given a node $v \in V$, let $H - v$ denote the hypergraph obtained from H by deleting each hyperedge containing v (either as a body node or a head node). We say that a node $v \in V$ is **reachable** from a set $Z \subseteq V$ in H if either $v \in Z$ or there exists a hyperedge (B, v) such that each node in B is reachable from Z in $H - v$. The **set of nodes reachable from** Z in H is denoted by $F_H(Z)$.

2.3. Pure Horn functions and directed hypergraphs

There is a natural one-to-one correspondence between pure Horn CNFs and directed hypergraphs. Namely, a CNF $\Phi = (V, \mathcal{C})$ and a hypergraph $H = (V, \mathcal{E})$ correspond to each other if $(B \to v) \in \mathcal{C}$ if and only if $(B, v) \in \mathcal{E}$. Let h be a pure Horn function, Φ be a pure Horn CNF representing h and H be the corresponding hypergraph. It is easy to see that $\mathcal{T}_h = \mathcal{T}_H$, $\mathcal{F}_h = \mathcal{F}_H$, $\mathcal{B}(\Phi) = \mathcal{B}(H)$, $\mathcal{H}(\Phi) = \mathcal{H}(H)$ and $F_h(Z) = F_H(Z)$ for every $Z \subseteq V$. Hence the problem of finding a body-minimal representation of h is equivalent to finding a hypergraph $H = (V, \mathcal{E})$ with $\mathcal{T}_H = \mathcal{T}_h$ and $|\mathcal{B}(H)|$ being minimal. For a given pure Horn CNF $\Phi = (V, \mathcal{C})$, we will denote the corresponding directed hypergraph by $H_\Phi = (V, \mathcal{E}_\Phi)$.

3. Body-minimal representation

Let $\mathcal{E}^*$ denote the set of all possible hyperedges on V, that is, $\mathcal{E}^* := \{(B, v) : \emptyset \neq B \subset V,\ v \in V \setminus B\}$. Let h be a pure Horn function. An hyperedge $(X, v) \in \mathcal{E}^*$ is called **valid** if it covers none of the true sets in $\mathcal{T}_h$. Observe that a hypergraph $H = (V, \mathcal{E})$ represents h if and only if it covers $\mathcal{F}_h$ and only has valid hyperedges. A true set Y is said to **separate** false sets X_1 and X_2 if $X_1 \cap X_2 \subseteq Y$ and either