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Decycling with a matching

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ABSTRACT

As a natural variant of the many decycling notions studied in graphs, we consider the problem to decide whether a given graph G has a matching M such that $G - M$ is a forest. We establish NP-completeness of this problem for 2-connected planar subcubic graphs, and describe polynomial time algorithms that also determine such a matching if it exists for graphs that are claw- and paw-free, P_5-free, chordal, and C_4-free distance hereditary.

1. Introduction

We consider finite, simple, and undirected graphs, and use standard notation and terminology.

Destroying all cycles of a given graph by removing vertices or edges is a classical theme. Clearly, the minimum number of edges of a connected graph of order n and size m whose removal destroys all cycles is exactly $m - n + 1$, and standard minimum spanning tree algorithms allow to solve even weighted optimization versions. Contrary to this, the minimum number of vertices whose removal destroys all cycles (or produces a tree) is a difficult parameter [2–4,6,8].

In the present paper we study a special case of the problem of destroying all cycles by removing only edges under the natural restriction that the graph formed by the removed edges has bounded maximum degree. In fact, we consider the apparently simple case when the removed edges are required to form a matching. Quite surprisingly, we show that the corresponding decision problem, that is, the problem to decide whether a given graph is the union of a tree and a matching, is already hard. Furthermore, we present efficient algorithms for a number of well-known graph classes.

For a set E of edges of a graph G, let $G - E$ be the graph with vertex set $V(G)$ and edge set $E(G) \setminus E$. If $G - E$ is a forest, then E is *decycling*. Let $\mathcal{FM}$ be the set of all graphs that have a decycling matching.

2. Results

The following lemma collects some basic observations concerning graphs that have a decycling matching.

Lemma 1. *Let G be a graph.*

(i) *If $G \in \mathcal{FM}$ is connected, then G has a matching M for which $G - M$ is a tree.*
(ii) *If $G \in \mathcal{FM}$, then $m(H) \le \left\lfloor \frac{3n(H)}{2} \right\rfloor - 1$ for every subgraph H of G.*
(iii) *If G is subcubic and connected, then $G \in \mathcal{FM}$ if and only if G has a spanning tree T such that all endvertices of T are of degree at most 2 in G.*

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Proof. (i) Let $G \in \mathcal{FM}$ be connected. Let M be a matching of G such that $G - M$ is a forest F with as few components as possible. Suppose, for a contradiction, that F is not connected. Since G is connected, M contains an edge e between different components of F. Now, $N = M \setminus \{e\}$ is a matching of G such that $G - N$ is a forest with less components than F, which implies a contradiction. Hence, F is a tree.

(ii) Let $G \in \mathcal{FM}$. Since $\mathcal{FM}$ is closed under taking subgraphs, it suffices to show $m(G) \leq \left\lfloor \frac{3n(G)}{2} \right\rfloor - 1$. Let M be a decycling matching of G. Clearly, $m(G) \leq m(G - M) + |M| \leq (n(G) - 1) + \left\lfloor \frac{n(G)}{2} \right\rfloor = \left\lfloor \frac{3n(G)}{2} \right\rfloor - 1$.

(iii) Let G be a connected subcubic graph. Clearly, we may assume that $n(G) \geq 3$.

First, suppose that $G \in \mathcal{FM}$. By (i), G has a matching M such that $G - M$ is a spanning tree T. If u is an endvertex of T, then $d_G(u) \leq d_T(u) + 1 \leq 2$, which implies that all endvertices of T have degree at most 2 in G.

Next, suppose that T is a spanning tree of G such that all endvertices of T are of degree at most 2 in G. Let $M = E(G) \setminus E(T)$. Clearly, M is decycling, and it remains to show that M is a matching. Suppose that M contains two edges incident with the same vertex u of G. This implies $d_T(u) \leq d_G(u) - 2 \leq 3 - 2 = 1$, that is, u is an endvertex of T. By the choice of T, we obtain $d_T(u) \leq d_G(u) - 2 \leq 2 - 2 = 0$, which is a contradiction. □

Lemma 1(iii) is the key observation for the following hardness result.

Theorem 2. *For a given 2-connected planar subcubic graph G, it is NP-complete to decide whether $G \in \mathcal{FM}$.*

Proof. The considered decision problem is clearly in NP. In order to show NP-completeness, we use [5] that deciding the existence of a Hamiltonian cycle for a given 3-connected planar cubic graph is NP-complete. In fact, the 3-connected planar cubic graphs G constructed by Garey et al. in [5] contain several edges that necessarily belong to every Hamiltonian cycle of G; regardless of whether such a cycle exists or not. Therefore, removing such an edge, their construction implies the NP-completeness of the following decision problem: *Given a 2-connected planar subcubic graph G with exactly two vertices u and v of degree 2, does G have a Hamiltonian path whose endvertices are u and v?*

Let G be a 2-connected planar subcubic graph with exactly two vertices u and v of degree 2. In order to complete the proof, it suffices to show that G has a Hamiltonian path whose endvertices are u and v if and only if $G \in \mathcal{FM}$. First, suppose that P is a Hamiltonian path of G whose endvertices are u and v. Clearly, P is a spanning tree of G such that all endvertices of P are of degree at most 2 in G. By Lemma 1(iii), this implies $G \in \mathcal{FM}$. Next, suppose that $G \in \mathcal{FM}$. By Lemma 1(iii), this implies that G has a spanning tree T such that all endvertices of T are of degree at most 2 in G. Since u and v are the only vertices of G of degree at most 2, this implies that T has exactly the two endvertices u and v. Hence, T is a Hamiltonian path of G whose endvertices are u and v. □

In order to enable suitable reductions, we now consider a slightly more general version of our decision problem.

ALLOWED DECYCLING MATCHING
Instance: A graph G and a set F of edges of G.
Task: Decide whether G has a decycling matching M that does not intersect F, and determine such a matching if it exists.

A matching M as in ALLOWED DECYCLING MATCHING is an *allowed decycling matching* of (G, F).

The *claw* $K_{1,3}$ and the *paw* $K_{1,3} + e$ are the unique graphs with degree sequences 1, 1, 1, 3 and 1, 2, 2, 3, respectively.

Theorem 3. ALLOWED DECYCLING MATCHING *can be solved in polynomial time for $\{K_{1,3}, K_{1,3} + e\}$-free graphs.*

Proof. Let G be a $\{K_{1,3}, K_{1,3} + e\}$-free graph and let F be a set of edges of G. Since (G, F) has an allowed decycling matching if and only if $(K, E(K) \cap F)$ has an allowed decycling matching for every component K of G, we may assume that G is connected.

The following claim is an immediate consequence of Lemma 1(ii).

Claim 1. *If G contains K_4 as an induced subgraph, then (G, F) has no allowed decycling matching.*

Claim 2. *If G has a vertex of degree at least 4, then (G, F) has no allowed decycling matching.*

Proof of Claim 2. Let u be a vertex of G with four neighbors v_1, v_2, v_3, and v_4. Since G is $\{K_{1,3}, K_{1,3} + e, K_4\}$-free, we may assume, by symmetry, that $v_1v_2, v_2v_3 \in E(G)$ and $v_1v_3 \notin E(G)$. Considering v_1, v_3, and v_4, this implies, by symmetry, that $v_3v_4 \in E(G)$. Considering the three triangles uv_1v_2u, uv_2v_3u, and uv_3v_4u, it follows that (G, F) does not have an allowed decycling matching. □

Since no endvertex of G lies on a cycle, we may assume that G has minimum degree at least 2. Since whether G is K_4-free and has maximum degree at most 3, can be tested in polynomial time, we may assume, by Claim 1 and Claim 2, that G is K_4-free and has maximum degree at most 3. If G does not have any vertex of degree 3, then G is a cycle, and (G, F) has an allowed decycling matching if and only if F does not contain all edges of G. Hence, we may assume that G has a vertex b of degree 3. Let $N_G(b) = \{a, c, d\}$. Since G is $\{K_{1,3}, K_{1,3} + e, K_4\}$-free, we may assume, by symmetry, that $ac, cd \in E(G)$ and $ad \notin E(G)$. Let $G' = (V(G) \setminus \{b, c\}, (E(G) \setminus \{ab, ac, bc, bd, cd\}) \cup \{ad\})$, and let $F' = (F \setminus \{ab, ac, bc, bd, cd\}) \cup \{ad\} \cup \{xa : x \in N_G(a) \setminus \{b, c\}\} \cup \{ad\} \cup \{yd : x \in N_G(d) \setminus \{b, c\}\}$.

Claim 3.

(i) *G' is $\{K_{1,3}, K_{1,3} + e\}$-free.*
(ii) *(G, F) has an allowed decycling matching if and only if*
- *$ab, cd \notin F$ or $ac, bd \notin F$, and*
- *(G', F') has an allowed decycling matching.*