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Source-wise round-trip spanners

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ABSTRACT

In this paper, we study a new type of graph spanners, called *source-wise round-trip spanners*. Given any source vertex set $S \subseteq V$ in a directed graph $G(V, E)$, such a spanner (with stretch k) has the property that the round-trip shortest path distance from any vertex $s \in S$ to any vertex $v \in V$ is at most k times of their round-trip distance in G. We present an algorithm to construct the source-wise round-trip spanners with stretch $(2k+\epsilon)$ and size $O((k^2/\epsilon)ns^{1/k}\log(nw))$ where $n = |V|, s = |S|$ and w is the maximum edge weight. The result out-performs the state-of-the-art traditional round-trip spanners when the source vertex set S has small cardinality, and at the same time, it matches the traditional spanners when S is the whole vertex set V.

1. Introduction

A graph spanner, which is a compact graph structure approximating pair-wise shortest path distances, has received intensive research interests since it was proposed in 1989 [15]. Formally, a *k-spanner* for an undirected graph $G(V, E)$ is a subgraph $(V, E' \subseteq E)$ such that for every $u, v \in V$, the distance between them is at most k times of their original distance in G. Here k is called *stretch* of the spanner. It is well known that for an undirected graph of n vertices, there exists a *trade-off* between the stretch and size (the number of edges), specifically, a $(2k-1)$-spanner of size $O(n^{1+1/k})$ [3]. The result is conjectured to be optimal if the Erdos's Conjecture is true [13]. A lot of research efforts were then devoted to (purely) additive spanners, where the distance between any pair of vertices is only larger than their distance in G by a (constant) additive term. Consider an undirected graph, there exists constructions of a 2-additive spanner of size $O(n^{3/2}\log^{1/2} n)$ [2] (the log factor is shaved in [12]), a 4-additive spanner of size $\tilde{O}(n^{7/5})$ [5], and a 6-additive spanner of size $O(n^{4/3})$ [4,20]. Very recently, Abboud et al. [1] showed that one cannot obtain any purely additive spanner using size $\Omega(n^{4/3})$. For many other research working on the spanners with mixed multiplicative and additive stretches readers are referred to [12,19,16,4].

Note that all of the above spanners are for undirected graphs. For *directed* graphs, it is even unmeaningful to study the spanner problem because of the well known $\Omega(n^2)$ size lower bound. Consider the bipartite graph (LHS, RHS, E) where each vertex $u \in LHS$ has edges $e \in E$ to all the vertices $v \in RHS$, even only to preserve connectivity, all edges in E must be preserved in any spanner. Therefore, instead of studying one-way distance between any two vertices, the round-trip distance between them, which is the sum of the two one-way distances with opposite direction, was studied *firstly* by Cowen and Wagner [8,9]. A *k-round-trip spanner* in a directed graph $G(V, E)$ is a subgraph $(V, E' \subseteq E)$ where, for each pair of vertices $u, v \in V$, the round-trip distance between them is at most k times of their round-trip distance in G. They worked on routing problems and implicitly generated (2^k-1)-round-trip spanners of size $\tilde{O}(n^{1+1/k})$. Later, Roditty et al. [18] improved the stretch and size trade-off of the round-

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trip spanners. In particular, their state-of-the-art algorithm generates, for any n-vertices graph with maximum edge weight w, a $(2k+\epsilon)$-round-trip spanner of size $O(\min\{(k^2/\epsilon)n^{1+1/k}\log(nw), (k/\epsilon)^2 n^{1+1/k}(\log n)^{2-1/k}\})$.

Another research thread works on constructing graph structures which approximate distances for some pairs of vertices, instead of all pairs. In this direction, Coppersmith and Elkin [7] is the first contribution by their work on exact source-wise spanners (called preservers). They showed that for any n-vertices undirected graph and s source vertices (or p pairs of vertices), there exists a source-wise preserver of size $O(\min\{n^{1/2}s^2, ns\})$ (a pair-wise preserver of size $O(\min\{np^{1/2}, n^{1/2}p\})$, respectively). Later, Roditty et al. [17] studied approximate source-wise distance spanners. The paper showed that for any undirected graph (V,E) and s source vertices S, there exists an algorithm to construct a source-wise $(2k-1)$-spanner of size $O(kns^{1/k})$ which can provide approximate distance guarantee with respect to $S \times V$ in $O(k|E|s^{1/k})$ expected time. Further studies along this direction focus on approximate source-wise spanners with additive stretch. See the constructions of source-wise spanners with stretch $2\log n, 6, 4$ and size $\tilde{O}(ns^{1/2})$, $\tilde{O}(n^{11/9}s^{2/9})$ and $O(n^{6/5}s^{1/5})$ in [10,14,14], respectively. An incomparable related work [11] is on embedding a metric into a simpler metric while approximating distances between a given terminal set and all other points.

In this paper, we study a new type of spanners: the *source-wise round-trip spanners* for directed graphs. For a directed graph $G(V,E)$ and source vertex set $S \subseteq V$, an *S-source-wise k-round-trip spanner* is a subgraph (V,E') where, for every $u \in S, v \in V$, the round-trip distance between u and v is at most k times of their round-trip distance in G. The source-wise round-trip spanners are natural extensions of the round-trip spanners and have a number of applications, to name a few, source-wise round-trip compact routing schemes, source-wise round-trip distance oracles, source-wise low distortion embeddings. Our result is an extension of Cohen [6] and Roditty et al. [18] and inspired by Roditty et al. [17], which is for source-wise (called source-restricted) approximate distance oracles. We adapt Cohen's construction of modified neighborhood covers [6] for undirected graphs to the setting of digraphs and extend it to source-wise setting. It is interesting to discover that Cohen's result [6] can be extended to source-wise setting. The new algorithm for the source-wise round-trip spanners and its rigorous proof provided in this paper are non-trivial.

Our contribution is summarized in the following theorem.

Theorem 1. *For every integer $k \geq 1$, real number $\epsilon > 0$, weighted directed graph $G(V,E)$ and source vertex set $S \subseteq V$, there exists an S-source-wise $(2k+\epsilon)$-round-trip spanner of size $O((k^2/\epsilon)ns^{1/k}\log(nw))$ where $n = |V|, s = |S|$ and w is the maximum edge weight.*

The result matches the traditional round-trip spanners when source vertex set $S = V$, and is the first result on (absolute size bounded) spanner structures for directed graphs in the past several years.

Algorithm 1 $cover(G(V,E),k,r,S)$.

Input: $G(V,E),k,r,S$
Output: C

```
C ← ∅; S_{k-1} ← S; V_{k-1} ← V;
for i ← k − 1 to 0 do
    S'_i ← sample(S_i, s^{-i/k});
    C ← C + {Ball_{V_i}(v, (i+1)r) | v ∈ S'_i};
    V_{i-1} ← V_i − ∪_{v∈S'_i} Ball_{V_i}(v, ir);
    S_{i-1} ← S_i − ∪_{v∈S'_i} Ball_{V_i}(v, ir);
end for
return C;
```

We present the road-map of this paper. In Section 2, we define all notations and symbols to be used in this paper. In Section 3, the details of algorithm and proof are presented. In Section 4, we conclude the paper and point out a few interesting future work.

2. Notation and definition

In a directed graph $G(V,E)$, a *(one-way) shortest path* from the vertex $u \in V$ to the vertex $v \in V$ is a path from u to v with the minimum *length* (sum of edge weight). A *round-trip shortest path* between u and v is a path from u to v and then back to u with the minimum length, which is called the *round-trip length* between u and v.

Definition 1. For a directed graph $G(V,E)$ and source vertex set $S \subseteq V$, an S-source-wise k-round-trip spanner is a subgraph (V,E') such that, for every $u \in S, v \in V$, the round-trip length between u and v is at most k times of their round-trip length in G.

A *(round-trip) ball $Ball_U(u,r)$* in G is a set of vertices whose round-trip length from u, in the subgraph induced by vertices in a vertex set U, is smaller than or equal to *radius r*. There may exist multiple round-trip shortest paths between any vertex pair and this does not invalidate the algorithm proposed in this paper.

3. Source-wise round-trip spanners

We would like to present the technical details in this section. We start by introducing a main building block of the algorithm for constructing spanners, namely constructing *covers*, and proving a key theorem. After that, we prove our main result stated in Theorem 1.

Definition 2. In a directed graph $G(V,E)$, a collection C of round-trip balls is a (k,r)-cover for vertex set $S \subseteq V$ if and only if each ball in C has radius at most kr, and for every $u \in S, v \in V$ such that their round-trip length is at most r, there is a ball $B \in C$ such that $u, v \in B$.

Algorithm 1 constructs (k,r)-covers given any digraph $G(V,E)$, two parameters k, r, and source vertex set S. At the beginning of the algorithm, cover C is initialized to $\emptyset$; S_{k-1} and V_{k-1} are initialized to S and V, respectively. The algorithm contains k iterations. We number an iteration by the value of the variable i, which decreases from $k-1$ to 0. In the ith iteration, each vertex of S_i is sampled to S'_i