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Some bounds on the generalised total chromatic number of degenerate graphs

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ABSTRACT

The total generalised colourings considered in this paper are colourings of the vertices and of the edges of graphs satisfying the following conditions:

- each set of vertices of the graph which receive the same colour induces an m-degenerate graph,
- each set of edges of the graph which receive the same colour induces an n-degenerate graph, and
- incident elements receive different colours.

Bounds for the least number of colours with which this can be done for all k-degenerate graphs are obtained.

1. Introduction

For graphs in general, we use the notation and terminology of [4]; for concepts related to (hereditary) graph properties we use the notation and terminology of [1]. Two particular graph properties to be used in the sequel are $\mathcal{O}$ and $\mathcal{O}_1$, where $\mathcal{O} = \{G \in \mathcal{I} : G$ is edgeless, i.e., $E(G) = \emptyset\}$ and $\mathcal{O}_k = \{G \in \mathcal{I} :$ each component of G has at most $k+1$ vertices$\}$ and $\mathcal{I}$ is the set of all graphs.

A graph G is called *k-degenerate* if the minimum degree $\delta(H) \le k$ for each induced subgraph H of G. The set of all k-degenerate graphs will be denoted by $\mathcal{D}_k$; it is a well-known additive induced hereditary graph property. k-degenerate graphs were introduced in [8] and they play an important role in the structure of hereditary properties of graphs (see e.g. [9,10]).

Let $\mathcal{P}$ and $\mathcal{Q}$ be graph properties and let $C = \{1, \dots, d\}$. If $G = (V, E)$ is a graph, then a function $c : V \cup E \to C$ is a *total* $(\mathcal{P}, \mathcal{Q})$*-colouring of G in d colours* if

(1) $G[\{c^{-1}(i)\} \cap V] \in \mathcal{P}$, for all $i \in C$,
(2) $G[\{c^{-1}(i)\} \cap E] \in \mathcal{Q}$, for all $i \in C$,
(3) if $e = vu \in E$ (with $v, u \in V$), then $c(v) \ne c(e)$ and $c(u) \ne c(e)$, i.e., no vertex receives the same colour as any edge incident to it.

The minimum number of colours needed in a total $(\mathcal{P}, \mathcal{Q})$-colouring of G is called the *total* $(\mathcal{P}, \mathcal{Q})$*-chromatic number* and is denoted by $\chi''_{\mathcal{P},\mathcal{Q}}(G)$ (see [2]). Clearly, when $\mathcal{P} = \mathcal{O}$ and $\mathcal{Q} = \mathcal{O}_1$, a total $(\mathcal{P}, \mathcal{Q})$-colouring of a graph G is nothing but a total colouring of G so that $\chi''_{\mathcal{O},\mathcal{O}_1}(G) = \chi''(G)$. This parameter is studied in [7] where it is shown that an s-degenerate graph has a total colouring with $\Delta + 1$ colours if the maximum degree Δ is sufficiently large.

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2. Motivation

To know the minimum number, or at least a bound for the minimum number of colours needed in a total $(\mathcal{P}, \mathcal{Q})$-colouring of a graph G, implies that we know in how many parts we can partition the vertices and the edges of the graph separately while imposing a restriction on the structure of each of these parts. In fact, we impose restrictions on the subgraph induced by each vertex part (by choosing a suitable $\mathcal{P}$) which are independent of the restrictions posed on the subgraph induced by each edge part (by choosing a suitable $\mathcal{Q}$). We shall now describe a possible application of this type of partition problem for networks which can be represented as graphs.

The theory of *wireless sensor networks* has become important in our modern day and age – see [3] for example. This is due to its many potential applications in process management, health care, environmental sensing, etc. Furthermore, this theory has interesting challenging theoretical problems.

A wireless sensor network (WSN) differs from a computer network in that it has limited capabilities of the sensors which could be caused by low energy sources or low computational capacity. Pairs of sensors typically communicate through designated channels. Because of potential collisions and interferences and a limited capability of the sensor involved, the number of communication channels linked to one sensor may be limited. In order to secure the communication sent through this network one can assign certificates to sensors. Again, due to limited computational capacity, it is only possible to use a fixed, but limited, number of certificates. On the other hand, the communication will be safer, if the same certificate is not used repeatedly. Such a network may therefore fail to function far away from maintenance engineers, deep under the sea or in outer space for example, if the limitation specifications imposed in its design are not strict enough for it to handle its task. It is therefore a reasonable option to structure its design in such a way that some parts of the WSN may still function optimally in such a situation. This option calls for labellings of the sensors and the communication channels during the design phase of the WSN in such a way that the subnetworks determined by sets of equally labelled sensors and equally labelled communication channels have suitable structural limitations to make parts of the network still functional.

This situation corresponds to a great extent to the problem we study in this paper: Think about the network as the graph G having as vertex set V the set of sensors and as edge set E the set of its communication channels. The limited number of communication channels linking one sensor to others may then be translated into a degree restriction for the vertices of the graph linking it to the graph G being k-degenerate for a suitable choice of k.

By determining for such a graph G its total $(\mathcal{D}_m, \mathcal{D}_n)$-chromatic number $\chi''_{\mathcal{D}_m,\mathcal{D}_n}(G)$, one obtains information on how many subnetworks of a similar kind, which could ensure that such subnetworks remain functional in case of a failure of the WSN, are needed. Condition (3) in the definition of a total $(\mathcal{P}, \mathcal{Q})$-colouring of a graph is perhaps not applicable to this situation. However, any upper bound on the number of colours needed can only be improved on by relaxing this condition.

Our particular choice of degree restrictions of the vertices of the subnetworks ensures stricter restrictions on its structural design. It was shown in [6] that WSN with degenerate topologies possesses specific properties that are very important for communication protocol design.

In this paper we then study, for positive integers m, n and k, the total $(\mathcal{D}_m, \mathcal{D}_n)$-chromatic number $\chi''_{\mathcal{D}_m,\mathcal{D}_n}(G)$ of a graph G with $G \in \mathcal{D}_k$.

3. The total colouring of degenerate graphs

In our first result we give an upper bound for $\chi''_{\mathcal{D}_m,\mathcal{D}_n}(G)$ for a graph $G \in \mathcal{D}_k$.

Theorem 1. *For every three positive integers m, n and k and for every $G \in \mathcal{D}_k$ we have* $\chi''_{\mathcal{D}_m,\mathcal{D}_n}(G) \le \max\left\{\left\lceil \frac{k+1}{m+1} \right\rceil, \left\lceil \frac{k}{n} \right\rceil + 2\right\}$.

Proof. Consider any three positive integers m, n and k. We denote, for convenience, the number $\max\left\{\left\lceil \frac{k+1}{m+1} \right\rceil, \left\lceil \frac{k}{n} \right\rceil + 2\right\}$ by x. The proof is by induction over the number of vertices of G. If G has only one vertex, the result holds since then $\chi''_{\mathcal{D}_m,\mathcal{D}_n}(G) = 1$ while $x \ge 3$ for all positive integers m, n and k.

Hence suppose the result holds for all k-degenerate graphs of order at most $p-1$ and let G be one of order p. Then G has a vertex of degree at most k; suppose v is such a vertex. Since $G - v$ is also k-degenerate, the induction hypothesis assures us that $\chi''_{\mathcal{D}_m,\mathcal{D}_n}(G - v) \le x$. Consider a total $(\mathcal{D}_m, \mathcal{D}_n)$ colouring of $G - v$ using x colours, which we will denote by $1, 2, \ldots, x$, and let, $W_1, W_2, \ldots, W_x$ be the colour classes into which the subset of $V(G - v)$ consisting of those vertices which are adjacent to v is partitioned by this colouring of the vertices of $G - v$.

We claim that at least one set W_i then contains at most m vertices. This is so since $x \ge \left\lceil \frac{k+1}{m+1} \right\rceil$ and hence $x \ge \frac{k+1}{m+1}$, i.e., $x(m+1) \ge k+1$. Hence, if each W_i contains at least $m+1$ vertices, then the degree of v is $|\bigcup_j W_j| \ge x(m+1) \ge k+1$ which contradicts the fact that the degree of v is at most k.

Therefore at least one of the W_i's, say W_x, contains at most m vertices: we can therefore colour v with x to complete the colouring of the vertices of G with x colours such that each colour class of vertices induces an m-degenerate graph as required.

In order to colour the edges incident to v without violating the incidence condition, each of the k edges incident to v must be coloured by a colour different from the colours of its endvertices; we shall call such a colour *admissible* at the edge. This means that we have $x-2$ possibilities for each edge incident to a vertex with colour different from x and $x-1$ possibilities for each edge of which both endvertices are coloured by x.

We shall show that we can assign colours to the edges incident to v is such a way that: