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Approximation schemes for the parametric knapsack problem

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ABSTRACT

We consider the (linear) parametric 0–1 knapsack problem in which the profits of the items are affine-linear functions of a real-valued parameter and the task is to compute a solution for all values of the parameter. For this problem, it is known that the piecewise linear convex function mapping the parameter to the optimal objective value of the corresponding instance (called the *optimal value function*) can have exponentially many breakpoints (points of slope change), which implies that every optimal algorithm for the problem must output a number of solutions that is exponential in the number of items.
We provide the first (parametric) polynomial time approximation scheme (PTAS) for the parametric 0–1 knapsack problem. Moreover, we exploit the connection between the parametric problem and the bicriteria problem in order to show that the parametric 0–1 knapsack problem admits a parametric FPTAS when the parameter is restricted to the positive real line and the slopes and intercepts of the affine-linear profit functions of the items are nonnegative. The method used to obtain this result applies to many linear parametric optimization problems and provides a general connection between bicriteria and linear parametric optimization problems.

1. Introduction

The knapsack problem is a well-studied combinatorial optimization problem with numerous applications. Given a knapsack capacity and a set of n items with different weights and profits, the task in the classical 0–1 knapsack problem is to select a subset of the items with maximum total profit subject to the constraint that the total weight of the selected items may not exceed the knapsack capacity. The 0–1 knapsack problem is NP-hard, but it admits a fully polynomial time approximation scheme (FPTAS) and can be solved exactly in pseudo-polynomial time by dynamic programming (cf. [1]).

The (linear) parametric 0–1 knapsack problem is a generalization of the 0–1 knapsack problem in which the profits of the items are affine-linear functions of a parameter $\lambda \in \mathbb{R}$. Here, the profit of each item i is given as $p_i = p_i(\lambda) = a_i + \lambda b_i$ with $a_i, b_i \in \mathbb{Z}$ and the problem can be written as

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} (a_i + \lambda b_i) \cdot x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i \cdot x_i \leq B \\ & x_i \in \{0,1\} \qquad \text{for } i = 1, \ldots, n \end{aligned}$$

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where $B \in \mathbb{N}$ denotes the knapsack capacity and $w_i \in \mathbb{N}_{>0}$ denotes the weight of item i. We note that, since positive as well as negative values a_i, b_i are allowed, some items might have negative profits even for positive values of λ. Moreover, the profits of some items might increase with increasing λ while the profits of other items might decrease.

For linear parametric optimization problems such as the parametric 0–1 knapsack problem, one is interested in obtaining optimal solutions of the problem for all values of λ on the real line (or within a given interval). Since the objective values of feasible solutions are affine-linear functions of λ, it is easy to see that such a collection of optimal solutions is given by a finite, increasing sequence of parameter values $-\infty = \lambda_0, \ldots, \lambda_{K+1} = +\infty$ together with an optimal solution for each interval $(-\infty, \lambda_1], [\lambda_1, \lambda_2], \ldots, [\lambda_{K-1}, \lambda_K], [\lambda_K, +\infty)$, where an *optimal* solution for an interval is a feasible solution that is optimal for all values of λ within the interval. The function mapping $\lambda \in \mathbb{R}$ to the optimal objective value of the given instance for this value of λ is called the *optimal value function* (or the *optimal cost curve*). The above structure of optimal solutions implies that the optimal value function is piecewise linear and convex (concave in case of a minimization problem) and its *breakpoints* (points of slope change) are exactly at the points $\lambda_1, \ldots, \lambda_K$ (assuming that K was chosen as small as possible). Thus, the number K of breakpoints is a natural measure of the complexity of the problem. For the parametric 0–1 knapsack problem, Carstensen [2] showed that the number of breakpoints can be exponential in the number n of items, which implies that every optimal algorithm for the parametric 0–1 knapsack problem has to output an exponential number of solutions. Moreover, she raised the question which approximations of the optimal value function are obtainable in polynomial time.

1.1. Previous work

Linear parametric optimization problems in which the objective values of feasible solutions are affine-linear functions of a real parameter are widely studied in the literature. Besides the parametric 0–1 knapsack problem studied here, examples include the parametric shortest path problem [3–6], the parametric minimum spanning tree problem [7], and the parametric minimum cost flow problem [2]. While the number of breakpoints in the optimal value function of the parametric minimum spanning tree problem is known to be polynomial in the input size of the problem [7], the optimal value function of the parametric minimum cost flow problem can have exponentially many breakpoints even when the slopes of the affine-linear functions are restricted to the set $\{0, 1\}$ [2] and the optimal value function of the parametric shortest path problem can have pseudo-exponentially many breakpoints ($n^{\Omega(\log n)}$ on graphs with n nodes) [5,6]. When the slopes of the affine-linear functions are integers in $\{-M, \ldots, M\}$ for some constant $M \in \mathbb{N}$, however, the number of breakpoints in the optimal value function of the parametric shortest path problem becomes polynomial [4]. In several variants of parametric maximum flow problems, it is known that the minimum cuts satisfy so-called "nesting properties", which imply that there are at most $n-1$ breakpoints in the optimal value function on graphs with n nodes [8–11]. Parametric versions of general linear programs, mixed integer programs, and nonlinear programs (where the most general cases consider also non-affine dependence on the parameter as well as constraints depending on the parameter) are also widely studied. For an extensive literature review on these problems, we refer to [12].

The parametric 0–1 knapsack problem first appeared in the work of Carstensen [2], who shows that the number of breakpoints in the optimal value function can be exponential in the number of items. This holds even when restricting λ to a compact interval on the positive real line $\mathbb{R}_{>0}$ with the property that all profits are positive within this interval. However, she also shows that the number of breakpoints in the optimal value function of any linear parametric binary integer program becomes linear in the number of variables when the slopes and/or intercepts of the affine-linear functions are integers in $\{-M, \ldots, M\}$ for some constant $M \in \mathbb{N}$. In particular, this implies that the number of breakpoints in the optimal value function of the parametric 0–1 knapsack problem becomes linear in the number of items under this assumption. Eben-Chaime [13] shows that the optimal value function of the parametric 0–1 knapsack problem (together with a corresponding optimal solution between any two breakpoints) can be computed in $\mathcal{O}(KnB)$, where K denotes the number of breakpoints. This is achieved by using a general method of Eisner and Severance [14], which can be used to solve any instance of a linear parametric optimization problem with K breakpoints in the optimal value function by solving the instance for $\mathcal{O}(K)$ fixed values of the parameter.

A problem closely related to the parametric 0–1 knapsack problem is the *inverse-parametric knapsack problem* [15], which consists of computing the smallest value of λ for which the optimal value function of the parametric 0–1 knapsack problem has value equal to a prespecified solution value. For this problem, pseudo-polynomial (exact) algorithms are provided by Burkard and Pferschy [15].

The parametric 0–1 knapsack problem is also closely related to the bicriteria 0–1 knapsack problem since it can be interpreted as the weighted sum scalarization of the bicriteria problem. Thus, the optimal solutions of the parametric problem on the positive real line are exactly the supported efficient solutions of the bicriteria problem. For the bicriteria and multicriteria 0–1 knapsack problem, where the profit of each item in all objective functions is assumed to be nonnegative, several (multicriteria) FPTAS are known [16–19], i.e., algorithms that, given $\epsilon > 0$, compute in time polynomial in the size of the input and $1/\epsilon$ a set of solutions that, for each efficient solution, contains a solution that is at most at a factor $(1-\epsilon)$ worse in all objective functions.

1.2. Our contribution

We show that the parametric 0–1 knapsack problem admits a (parametric) polynomial time approximation scheme (PTAS). This means that, for any given $\epsilon > 0$, there