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Area-universal drawings of biconnected outerplane graphs

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ABSTRACT

Contact graph representation is a classical graph drawing style where vertices are represented by geometric objects such that edges correspond to contacts between the objects. Contact graph representations using axis-aligned rectilinear polygons are well-investigated. On the other hand, only a scarcity of results and techniques are available for cases using polygons that are not necessarily rectilinear. In this paper, we investigate a type of contact graph representations (named t-TkR) using k-sided convex polygons with their boundaries being t-sided. Given a biconnected outerplane graph, we present a clean necessary and sufficient condition for the graph to admit a t-TkR. We give a linear time algorithm for constructing an area-universal 3-T4R of a given biconnected outerplane graph, which is of interest since most of the previous results on area-universal drawings are with respect to rectilinear settings.

1. Introduction

A *contact graph representation* of a planar graph is a drawing in which vertices are represented by interior-disjoint geometric objects such that edges correspond to contacts between those objects. Following Koebe's circle packing theorem that every planar graph can be drawn as touching circles, a variety of contact graph representations have been proposed and studied in the literature over the years, see, e.g., [3–5,8–10].

Motivated by various applications in floor-planning, cartographic design, and data visualization, *rectilinear duals*, in which all vertices are represented by axis-aligned rectilinear polygons such that the drawing forms a tiling of a rectangle, have received extensive investigation in both VLSI design and graph drawing communities. The *polygonal complexity* of a rectilinear dual is defined as the maximum number of sides of any polygon in the drawing. A rectilinear dual of a graph G is called *area-universal* if it can realize any area-assignment $f : V(G) \to \mathbb{R}_{>0}$ in the sense that for every $v \in V(G)$, the corresponding polygon has area $f(v)$. Designing algorithms for constructing area-universal rectilinear duals of low polygonal complexity has been the focus of a number of recent results (see [3] and its citations).

In practice, it is common to encounter objects displayed as polygons that are not necessarily rectilinear. In contrast to the relatively well-studied rectilinear cases, only a scarcity of results and methods are available for tackling cases for polygons that are not necessarily rectilinear.

To extend the study of rectilinear duals to broader settings, the drawing style *convex polygonal dual* is proposed as a convex polygonal analogue of rectilinear duals [4]. Formally, a convex polygonal dual is a contact representation of a graph in which vertices are represented by convex polygons such that the drawing forms a tiling of a convex polygon. A drawing is called *k-sided* if each vertex is represented by a polygon of at most k sides in the drawing.

Our interests in this paper focus on biconnected outerplane graphs having *(t, k)-touching convex polygon representations*, which are k-sided convex polygonal duals with

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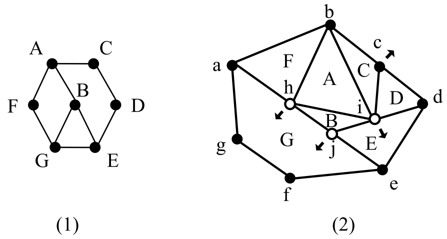

Fig. 1. A graph G and its convex polygonal dual G^d.

their boundary polygons being t-sided. We abbreviate such a representation as t-TkR. For instance, Fig. 1(2) is a 6-T4R.

The purpose of this paper is to study convex polygonal duals for biconnected outerplane graphs. For a biconnected outerplane graph G, we present:

1. A clean necessary and sufficient condition for the existence of a t-TkR, for $k > 3$: G admits a t-TkR iff $3 \le t \le (k-1)|V(G)| - |E(G)| + 1$.
2. A simple linear time algorithm for constructing an area-universal 3-T4R of G.

Related work. The study of representing graphs by touching triangles was initiated in [9]. It is known that triconnected cubic plane graphs [10] and strongly outerplane graphs [8] admit 3-T3R. Plane graphs having straight-line drawings with only triangular faces have been characterized by flat-angle assignments [1] and Schnyder labellings [2]. For contact representations of convex polygons, it was shown in [5] that 6-sided polygons are necessary and sufficient for plane graphs if holes are allowed. In [4], it was shown that convex polygonal duals can be defined in Monadic Second-Order Logic, yielding fixed-parameter tractability results for checking whether various plane graphs admit convex polygonal duals. For area-universal drawings, 12-sided polygons are known to be necessary and sufficient for rectilinear duals [3]. To our best knowledge, the work of [7] on table cartograms is the only result on area-universal drawings in a non-rectilinear setting.

2. Preliminaries

A graph is *planar* iff it can be drawn in the Euclidean plane without edge crossings. A *plane graph* is a planar graph with a fixed combinatorial embedding and a designated outer face. We write $f_O(G)$ to denote the outer face of a plane graph $G = (V, E)$. All the faces other than $f_O(G)$ are called *inner faces*. A vertex (or an edge) is called *boundary* if it is located in $f_O(G)$; otherwise, it is *non-boundary*.

An *outerplanar graph* is a planar graph with a planar embedding in which all vertices belong to the outer face. An outerplanar graph with such an embedding is called an *outerplane graph*. A graph is *biconnected* if removing any single vertex does not render the graph disconnected.

We write $\overline{xy}$ to denote a side of a polygon whose two end points are x and y. See Fig. 1 for an example of a convex polygonal dual. Note that convex polygon G in Fig. 1(2) has four sides, namely, $\overline{ag}$, $\overline{gf}$, $\overline{fe}$ and $\overline{ea}$. Note that the side $\overline{ea}$ consists of three segments (i.e., edges) (e, j), (j, h) and (h, a).

In a convex polygonal dual G^d, *junction points* are points that are endpoints of some segments in the drawing. For convenience, we write $BJ(G^d)$ and $NJ(G^d)$ to denote the sets of boundary and non-boundary junction points of G^d, respectively. In Fig. 1(2), there are 10 junction points with $BJ(G^d) = \{a, b, c, d, e, f, g\}$ and $NJ(G^d) = \{h, i, j\}$. Note that c is interior to one side $\overline{bd}$ of the boundary polygon. The arrows in the drawing indicate 180° angles.

3. Convex polygonal duals of biconnected outerplane graphs

With respect to a t-TkR of a biconnected outerplane graph, we first prove the following lemma which gives an upper bound on the number of sides of the boundary polygon (i.e., t):

Lemma 1. *Let G be a biconnected outerplane graph. If G admits a t-TkR, then $3 \le t \le (k-1)|V(G)| - |E(G)| + 1$. Moreover, the equality $t = (k-1)|V(G)| - |E(G)| + 1$ holds iff in the drawing,*

(1) *each polygon is exactly k-sided, and*
(2) *each non-boundary junction point is interior to a side of a polygon.*

Proof. The $t \ge 3$ is obvious since a polygon must have at least 3 sides. Let N be the total number of polygon corners in the t-TkR, say G^d, of G. For convenience, G^d is also referred to as a drawing. As each vertex in $V(G)$ corresponds to a polygon (of at most k sides) in G^d, $N \le k|V(G)|$. Since G is a biconnected outerplane graph, each polygon must intersect the boundary of the drawing in one connected path or a point; otherwise, the vertex corresponding to that polygon will be a cut-vertex in G-violating the assumption of G being biconnected. Since a path of s sides has $s+1$ corners, when a k-sided polygon contains s sides on the boundary of the drawing, it has exactly k-s-1 corners located not along the boundary of the drawing G^d.

Let $N = N_O + N_I$, where N_O denotes the total number of corners located along the boundary of the drawing (i.e., corners associated with boundary junction points), and N_I denotes the total numbers of corners located in the interior of the drawing (i.e., corners associated with non-boundary junction points). First, we show that $N_O \ge |V(G)| + t$. To see this, suppose N_v is the number of sides on the boundary of the drawing that intersect with the polygon corresponding to v. Note that a side can intersect with more than one polygon. For instance, in Fig. 1(2) $N_C = N_D = 1$ and the polygons corresponding to vertices C and D intersect with side $\overline{bcd}$. In view of above, $N_O = \sum_{v \in V(G)}(N_v + 1) = \sum_{v \in V(G)} N_v + |V(G)| \ge t + |V(G)|$.

For N_I, we argue that $N_I \ge \sum_{p \in NJ(G^d)} \deg(p) - |NJ(G^d)|$. Since each junction point can be associated with at most one 180° angle, the number of 180° angles at non-boundary junction points is at most $|NJ(G^d)|$. Hence the above inequality holds. As we note that each of $NJ(G^d)$ corresponds to an inner face of G, according to Euler's formula, $|NJ(G^d)| = |E(G)| - |V(G)| + 1$. For the