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Periodicity in rectangular arrays

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ABSTRACT

We discuss several two-dimensional generalizations of the familiar Lyndon–Schützenberger periodicity theorem for words. We consider the notion of primitive array (as one that cannot be expressed as the repetition of smaller arrays). We count the number of $m \times n$ arrays that are primitive. Finally, we show that one can test primitivity and compute the primitive root of an array in linear time.

1. Introduction

Let Σ be a finite alphabet. One very general version of the famous Lyndon–Schützenberger theorem [18] can be stated as follows:

Theorem 1. *Let $x, y \in \Sigma^+$. Then the following five conditions are equivalent:*

(1) $xy = yx$;
(2) There exist $z \in \Sigma^+$ and integers $k, \ell > 0$ such that $x = z^k$ and $y = z^\ell$;
(3) There exist integers $i, j > 0$ such that $x^i = y^j$;
(4) There exist integers $r, s > 0$ such that $x^r y^s = y^s x^r$;
(5) $x\{x, y\}^* \cap y\{x, y\}^* \neq \emptyset$.

Proof. For a proof of the equivalence of (1), (2), and (3), see, for example [23, Theorem 2.3.3].

Condition (5) is essentially the "defect theorem"; see, for example, [17, Cor. 1.2.6].

For completeness, we now demonstrate the equivalence of (4) and (5) to each other and to conditions (1)–(3):

(3) $\Longrightarrow$ (4): If $x^i = y^j$, then we immediately have $x^r y^s = y^s x^r$ with $r = i$ and $s = j$.

(4) $\Longrightarrow$ (5): Let $z = x^r y^s$. Then by (4) we have $z = y^s x^r$. So $z = xx^{r-1}y^s$ and $z = yy^{s-1}x^r$. Thus $z \in x\{x, y\}^*$ and $z \in y\{x, y\}^*$. So $x\{x, y\}^* \cap y\{x, y\}^* \neq \emptyset$.

(5) $\Longrightarrow$ (1): By induction on the length of $|xy|$. The base case is $|xy| = 2$. More generally, if $|x| = |y|$ then clearly (5) implies $x = y$ and so (1) holds. Otherwise without loss of generality $|x| < |y|$. Suppose $z \in x\{x, y\}^*$ and $z \in y\{x, y\}^*$. Then x is a proper prefix of y, so write $y = xw$ for a nonempty word w. Then z has prefix xx and also prefix xw. Thus $x^{-1}z \in x\{x, w\}^*$ and $x^{-1}z \in w\{x, w\}^*$, where by $x^{-1}z$ we mean remove the prefix x from z. So $x\{x, w\}^* \cap w\{x, w\}^* \neq \emptyset$, so by induction (1) holds for x and w, so $xw = wx$. Then $yx = (xw)x = x(wx) = xy$. □

A nonempty word z is *primitive* if it cannot be written in the form $z = w^e$ for a word w and an integer $e \geq 2$. We will need the following fact (e.g., [17, Prop. 1.3.1] or [23, Thm. 2.3.4]):

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Fact 2. Given a nonempty word x, the shortest word z such that $x = z^i$ for some integer $i \geq 1$ is primitive. It is called the *primitive root* of x, and is unique.

In this paper we consider generalizations of the Lyndon–Schützenberger theorem and the notion of primitivity to two-dimensional rectangular arrays (sometimes called *pictures* in the literature). For more about basic operations on these arrays, see, for example, [11].

2. Rectangular arrays

By $\Sigma^{m\times n}$ we mean the set of all $m \times n$ rectangular arrays A of elements chosen from the alphabet Σ. Our arrays are indexed starting at position 0, so that $A[0,0]$ is the element in the upper left corner of the array A. We use the notation $A[i..j, k..\ell]$ to denote the rectangular subarray with rows i through j and columns k through ℓ. If $A \in \Sigma^{m\times n}$, then $|A| = mn$ is the number of entries in A.

We also generalize the notion of powers as follows. If $A \in \Sigma^{m\times n}$ then by $A^{p\times q}$ we mean the array constructed by repeating A pq times, in p rows and q columns. More formally $A^{p\times q}$ is the $pm \times qn$ array B satisfying $B[i, j] = A[i \bmod m, j \bmod n]$ for $0 \leq i < pm$ and $0 \leq j < qn$. For example, if

$$A = \begin{bmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} \\ \mathtt{d} & \mathtt{e} & \mathtt{f} \end{bmatrix},$$

then

$$A^{2\times 3} = \begin{bmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{b} & \mathtt{c} \\ \mathtt{d} & \mathtt{e} & \mathtt{f} & \mathtt{d} & \mathtt{e} & \mathtt{f} & \mathtt{d} & \mathtt{e} & \mathtt{f} \\ \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{b} & \mathtt{c} \\ \mathtt{d} & \mathtt{e} & \mathtt{f} & \mathtt{d} & \mathtt{e} & \mathtt{f} & \mathtt{d} & \mathtt{e} & \mathtt{f} \end{bmatrix}.$$

We can also generalize the notation of concatenation of arrays, but now there are two annoyances: first, we need to decide if we are concatenating horizontally or vertically, and second, to obtain a rectangular array, we need to insist on a matching of dimensions.

If A is an $m \times n_1$ array and B is an $m \times n_2$ array, then by $A \oplus B$ we mean the $m \times (n_1 + n_2)$ array obtained by placing B to the right of A.

If A is an $m_1 \times n$ array and B is an $m_2 \times n$ array, then by $A \ominus B$ we mean the $(m_1 + m_2) \times n$ array obtained by placing B underneath A.

3. Generalizing the Lyndon–Schützenberger theorem

We now state our first generalization of the Lyndon–Schützenberger theorem to two-dimensional arrays, which generalizes claims (2), (3), and (4) of Theorem 1.

Theorem 3. *Let A and B be nonempty arrays. Then the following three conditions are equivalent:*

(a) There exist positive integers p_1, p_2, q_1, q_2 such that $A^{p_1\times q_1} = B^{p_2\times q_2}$.

(b) There exist a nonempty array C and positive integers r_1, r_2, s_1, s_2 such that $A = C^{r_1\times s_1}$ and $B = C^{r_2\times s_2}$.

(c) There exist positive integers t_1, t_2, u_1, u_2 such that $A^{t_1\times u_1} \circ B^{t_2\times u_2} = B^{t_2\times u_2} \circ A^{t_1\times u_1}$ where $\circ$ can be either $\oplus$ or $\ominus$.

Proof.

(a) $\Longrightarrow$ (b). Let A be an array in $\Sigma^{m_1\times n_1}$ and B be an array in $\Sigma^{m_2\times n_2}$ such that $A^{p_1\times q_1} = B^{p_2\times q_2}$. By dimensional considerations we have $m_1p_1 = m_2p_2$ and $n_1q_1 = n_2q_2$. Define $P = A^{p_1\times 1}$ and $Q = B^{p_2\times 1}$. We have $P^{1\times q_1} = Q^{1\times q_2}$. Viewing P and Q as words over $\Sigma^{m_1p_1\times 1}$ and considering horizontal concatenation, this can be written $P^{q_1} = Q^{q_2}$. By Theorem 1 there exist a word R over $\Sigma^{m_1p_1\times 1}$ and integers s_1, s_2 such that $P = R^{1\times s_1}$ and $Q = R^{1\times s_2}$. Let r denote the number of columns of R and let $S = A[0\ldots m_1 - 1, 0\ldots r-1]$ and $T = B[0\ldots m_2 - 1, 0\ldots r-1]$. Observe $A = S^{1\times s_1}$ and $B = T^{1\times s_2}$. Considering the r first columns of P and Q, we have $S^{p_1\times 1} = T^{p_2\times 1}$. Viewing S and T as words over $\Sigma^{1\times r}$ and considering vertical concatenation, we can rewrite $S^{p_1} = T^{p_2}$. By Theorem 1 again, there exist a word C over $\Sigma^{1\times r}$ and integers r_1, r_2 such that $S = C^{r_1\times 1}$ and $T = C^{r_2\times 1}$. Therefore, $A = C^{r_1\times s_1}$ and $B = C^{r_2\times s_2}$.

(b) $\Longrightarrow$ (c). Without loss of generality, assume that the concatenation operation is $\oplus$. Let us recall that $A = C^{r_1\times s_1}$ and $B = C^{r_2\times s_2}$. Take $t_1 = r_2$ and $t_2 = r_1$ and $u_1 = s_2$ and $u_2 = s_1$. Then we have

$$\begin{aligned} & A^{t_1\times u_1} \oplus B^{t_2\times u_2} \\ &= C^{r_1t_1\times s_1u_1} \oplus C^{r_2t_2\times s_2u_2} \\ &= C^{r_1t_1\times(s_1u_1+s_2u_2)} \qquad \text{(Observe that } r_1t_1 = r_2t_2\text{)} \\ &= C^{r_2t_2\times s_2u_2} \oplus C^{r_1t_1\times s_1u_1} \\ &= B^{t_2\times u_2} \oplus A^{t_1\times u_1}. \end{aligned}$$

(c) $\Longrightarrow$ (a). Without loss of generality, assume that the concatenation operation is $\oplus$. Assume the existence of positive integers t_1, t_2, u_1, u_2 such that

$$A^{t_1\times u_1} \oplus B^{t_2\times u_2} = B^{t_2\times u_2} \oplus A^{t_1\times u_1}.$$

An immediate induction allows to prove that for all positive integers i and j,

$$A^{t_1\times iu_1} \oplus B^{t_2\times ju_2} = B^{t_2\times ju_2} \oplus A^{t_1\times iu_1}. \qquad (1)$$

Assume that A is in $\Sigma^{m_1\times n_1}$ and B is in $\Sigma^{m_2\times n_2}$. For $i = n_2u_2$ and $j = n_1u_1$, we get $iu_1n_1 = ju_2n_2$. Then, by considering the first iu_1n_1 columns of the array defined in (1), we get $A^{t_1\times iu_1} = B^{t_2\times ju_2}$. □

Note that generalizing condition (1) of Theorem 1 requires considering arrays with the same number of rows or same number of columns. Hence the next result is a direct consequence of the previous theorem.

Corollary 4. *Let A, B be nonempty rectangular arrays. Then*
(a) if A and B have the same number of rows, $A \oplus B = B \oplus A$ if and only if there exist a nonempty array C and integers $e, f \geq 1$ such that $A = C^{1\times e}$ and $B = C^{1\times f}$;
(b) if A and B have the same number of columns, $A \ominus B = B \ominus A$ if and only if there exist a nonempty array C and integers $e, f \geq 1$ such that $A = C^{e\times 1}$ and $B = C^{f\times 1}$.