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Non-cooperative capacitated facility location games ☆

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ABSTRACT

We study capacitated facility location games, where players control terminals and need to connect each one to a facility from a set of possible locations. There are opening costs and capacity restrictions for each facility. Also, there are connection costs for each pair of facility and terminal if such facility attends this terminal. This problem has several applications, especially in distributed scenarios where a central authority is too expensive or even infeasible to exist. In this paper, we analyze and present new results concerning the existence of equilibria, Price of Anarchy (PoA), and Stability (PoS) for metric and non-metric versions of this game. We prove unbounded PoA and PoS for some versions of the game, even when sequential versions are considered. For metric variants, we prove that sequentiality leads to bounded PoA and PoS.

1. Introduction and notation

In game theory, a *non-cooperative game* is a scenario where players or agents choose strategies independently and try to either minimize their costs or maximize their utility. For each player i there is a set A_i of actions that it can choose to play. A pure strategy S_i consists of one action from A_i, while a mixed strategy corresponds to a probability distribution over A_i. In this paper we assume players pick pure strategies unless mentioned otherwise. A strategy profile, denoted by $S = (S_1, \ldots, S_k)$, corresponds to a solution of the game where each player $i = 1, \ldots, k$ chooses a strategy S_i.

We consider capacitated facility location games with and without a cost sharing scheme, which means that the cost to open a facility can be divided equally among all terminals connected to it (fair cost sharing) or it can be divided without any rules (no cost sharing rules).

Now we give formal definitions of the games considered in this paper. Let $G = (T \cup F, T \times F)$ be a bipartite graph, with vertex sets F of n facilities and T of m terminals. Each facility $f \in F$ has an opening cost c_f and a capacity u_f indicating how many terminals can be connected to f at any given time. Furthermore, there are connection costs d_{tf} for each pair terminal $t \in T$ and facility $f \in F$. In games with general distance costs, some connections (t, f) should be avoided in any solution, because they don't exist for example. In this case we assume they have a prohibitively large constant cost $\mathcal{U}_d$. When a connection is not shown, it is assumed that it has a cost equal to $\mathcal{U}_d$, unless mentioned otherwise. Let $K = [1, \ldots, k]$ be the set of players. Each player i controls a subset of terminals $T_i \subseteq T$ forming a partition of T, and each terminal must be connected to exactly one opened facility. When a player controls only a single terminal he is denominated a *singleton* player.

In the *Capacitated Facility Location Game with no cost sharing rules (CFLG)*, the set of actions A_i of player i is composed by tuples $(\mathcal{F}_i, p_i^c)$ where $\mathcal{F}_i : T_i \to F$ maps each

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terminal i controls to a facility, and $p_i^c : F \to \mathbb{R}_0^+$ maps the amount i pays to open facility f if some of its terminals is connected to it. Given some strategy S_i chosen by i, we simplify the notation by writing $(t, f) \in S_i$ to represent each connection i choose to its terminals. Likewise we write $f \in S_i$ to represent each facility where some terminal of i is connected to. The total amount paid by player i in strategy profile S is

$$p_i(S) = \sum_{f \in S_i} p_i^c(f) + \sum_{(t,f) \in S_i} d_{tf} \ .$$

Let $p^c(f) = \sum_{i=1}^{k} p_i^c(f)$ be the total paid by players for a facility f. If $p^c(f)$ is greater than or equal to the cost c_f, then the facility f is considered opened. Each player tries to minimize his payment. We denote the number of players connected to a facility f in a solution S by $x_f(S) = |\{1 \leq i \leq k : f \in S_i\}|$.

Solutions where there are more terminals connected to some facility f than its capacity u_f should be avoided. Moreover, solutions where terminals do not pay enough to open the facility they are connected to, should also be prevented. To avoid such solutions we add a prohibitively large constant cost $\mathcal{U}_c$ to the payment of terminals in such situations. For a player i, if there is a connection $(t, f) \in S_i$ where $p^c(f) < c_f$ or the number of players connected to f is greater than its capacity ($x_f(S) > u_f$), a prohibitively large constant cost $\mathcal{U}_c > \mathcal{U}_d$ is added to the total amount paid by i, i.e., he pays $p_i(S) + \mathcal{U}_c$.

For *Capacitated Facility Location Games with Fair-Cost sharing (CFLG-FC)*, a player i chooses a strategy $S_i \subset T_i \times F$ such that in S_i each terminal controlled by i is connected to exactly one facility. Let $S = (S_1, ..., S_k)$ be a strategy profile. Each player tries to minimize his own payment

$$p_i(S) = \sum_{f \in S_i} \frac{c_f}{x_f(S)} + \sum_{(t,f) \in S_i} d_{tf} \ ,$$

where $x_f(S) = |\{1 \leq i \leq k : f \in S_i\}|$ is the number of players using facility f in strategy profile S. Again, to ensure that capacity restrictions are respected, if a player i in the solution S has one of his terminals connected to f where $x_f(S) > u_f$, then a prohibitively large constant cost $\mathcal{U}_c$ is added to the payment of player i, i.e., he pays $p_i(S) + \mathcal{U}_c$.

Let $S_{-i} = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ be a strategy profile S without i's strategy, so that we can write $S = (S_i, S_{-i})$. *Pure Nash Equilibria* (PNE) are strategy profiles where no player can decrease his own costs by unilaterally changing his strategy, i.e., S is a PNE if for each player i, $p_i(S_i, S_{-i}) \leq p_i(S'_i, S_{-i})$ for all $S'_i \in A_i$.

The *social cost* is a function mapping a strategy profile to a real number, indicating a measure of the total cost of a game. We use the expression $f \in S$ to represent all facilities connected to a terminal in a strategy profile S, and $(t, f) \in S$ to represent all connections established in S. The social cost of a strategy profile S is defined for this game as the sum of all player payments, i.e.

$$C(S) = \sum_{i \in K} p_i(S) = \sum_{f \in S} c_f + \sum_{(t,f) \in S} d_{tf} \ . \tag{1}$$

Two of the most important concepts for efficiency analysis are the *Price of Anarchy* (PoA) and the *Price of Stability* (PoS). The PoA is the ratio between a Nash equilibrium with worst possible social cost and the strategy profile with optimal social cost, while the PoS is the ratio between the best possible Nash equilibrium and the social optimum. In the facility location games analyzed in this paper, the optimal social cost is the cost of an optimum solution for the corresponding optimization version of the problem.

Solution concepts such as pure Nash equilibria usually assume that players choose strategies simultaneously. This requirement can lead to unintuitive equilibria for facility location games where players choose to open expensive facilities when cheaper ones are also available. A possibility to take sequential movements in consideration is to analyze these games as *sequential games* [1,2]. In these games, players choose their strategies in a predefined arbitrary order. In the sequential facility location games considered in this paper, we assume each player $i \in [1, k]$ chooses a strategy only once given all strategies chosen by players before, so player 1 chooses first then player 2, and so on until player k.

An alternative solution concept that aims to better represent such scenarios is *Subgame Perfect Equilibrium* (SPE). Sequential games are usually represented as extensive form games, in the form of a game tree where each node represents a player and edges represent possible actions from the player on that node. SPE is defined as a strategy profile which is a PNE in every subgame of this game tree, so a SPE is also a PNE for the entire game. The *Sequential Price of Anarchy* (SPoA) is defined as the ratio between the cost of the worst subgame perfect equilibrium and the optimal social cost, while the *Sequential Price of Stability* (SPoS) is the ratio between the best SPE and the optimal social cost. One important aspect of such games is that they always have a SPE which can be computed using a method called backward induction. For further details on SPE and extensive form games see Chapter 4 of [2].

2. Related work and contributions

Facility location has been analyzed in a game-theoretic perspective from several directions. From mechanism design and strategy-proof mechanisms [3–5], to cooperative facility location [6] and valid utility games [7]. When there is competition between facilities to dominate markets, facilities may be modeled as players in a game-theoretic setting. These facility location problems are described as competitive location [8], with several relevant results in the literature [9–11].

In this paper we consider only the case where players control terminals, where each one requires a connection to an open facility. These facility location games can be viewed as connection games where every player starts from a single source vertex on a two-layered directed graph, and therefore multiple results for the uncapacitated versions of facility location games can be adapted from connection and network creation games.

For uncapacitated facility location games with fair cost sharing rules, most results can be adapted from cost sharing games and network design [12]. The PoA and PoS can be proven to be k and $H_k = \Theta(\log k)$, respectively, the same bounds obtained for network design [12].