Information Processing Letters 119 (2017) 9–13

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

PTAS for minimum k-path vertex cover in ball graph

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ARTICLE INFO

Article history:
Received 15 February 2016
Received in revised form 15 November 2016
Accepted 16 November 2016
Available online 21 November 2016
Communicated by Ł. Kowalik

Keywords:
Approximation algorithms
k-path vertex cover
Ball graph
PTAS

ABSTRACT

A vertex set F is a k-path vertex cover (VCP_k) of graph G if every path of G on k vertices contains at least one vertex from F. A graph G is a d-dimensional ball graph if each vertex of G corresponds to a ball in $\mathbb{R}^d$, two vertices are adjacent in G if and only if their corresponding balls have nonempty intersection. The heterogeneity of a ball graph is defined to be $r_{\max}/r_{\min}$, where $r_{\max}$ and $r_{\min}$ are the largest radius and the smallest radius of those balls, respectively. In this paper, we present a PTAS for the minimum VCP_k problem in a ball graph whose heterogeneity is bounded by a constant.

1. Introduction

Suppose $G=(V,E)$ is a graph, where V and E are the vertex set and the edge set, respectively. A path on k vertices is called a *k-path*. A vertex subset $F \subseteq V$ is a *k-path vertex cover* (abbreviated as VCP_k) of G if every k-path of G is *hit* by F, that is, every k-path of G contains at least one vertex from F. In other words, F is VCP_k of G if and only if $G-F$ does not contain k-path (or we say that $G-F$ is P_k-free). In the *minimum* VCP_k *problem* ($MVCP_k$), the goal is to find a VCP_k of the minimum cardinality. In particular, the minimum VCP_2 problem is exactly the minimum vertex cover problem, which is one of Karp's 21 NP-hard problems.

This paper studies approximation algorithm for $MVCP_k$ in ball graphs. In a *d-dimensional ball graph* G, each vertex corresponds to a ball in $\mathbb{R}^d$, two vertices are adjacent in G if and only if the two balls corresponding to them have nonempty intersection. In the following, we will not distinguish a ball and the vertex that it represents. The set of balls corresponding to $V(G)$ is called a *geometric representation* of G. In particular, a 2-dimensional ball graph is exactly a *disk graph* which is widely studied in the literature. A disk graph in which all disks have the same radii is a *unit disk graph*.

For a point $c \in \mathbb{R}^d$ and a positive real number r, we shall use $Ball(c,r)$ to denote the ball with center c and radius r. On the other hand, given a ball B in $\mathbb{R}^d$, we shall use c_B and r_B to denote its center and its radius, respectively. For a collection $\mathcal{B}$ of balls in $\mathbb{R}^d$, let $r_{\max}(\mathcal{B}) = \max\{r_B : B \in \mathcal{B}\}$ and $r_{\min}(\mathcal{B}) = \min\{r_B : B \in \mathcal{B}\}$. Call $h(\mathcal{B}) = r_{\max}(\mathcal{B})/r_{\min}(\mathcal{B})$ the *heterogeneity* of $\mathcal{B}$. When the collection $\mathcal{B}$ is clear, we shall omit $\mathcal{B}$ in the above notation for simplicity.

In this paper, for fixed integer k, we present a polynomial-time approximation scheme (PTAS) for the minimum VCP_k problem on a ball graph whose heterogeneity is bounded by a constant, that is, the performance ratio can achieve $(1+\varepsilon)$ for any fixed real number $\varepsilon > 0$.

2. Related works

The concept of k-path vertex cover first appeared in [20] with a background in security protocol design. In a

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http://dx.doi.org/10.1016/j.ipl.2016.11.003
0020-0190/

k-generalized Canvas scheme, data integrity is guaranteed by protecting some vertices such that each k-path has at least one protected vertex. Since protecting a vertex incurs more cost, it is desirable to minimize the number of protected vertices. This is exactly the minimum VCP$_k$ problem.

It was proved by Brešar et al. [2] that MVCP$_k$ is NP-hard for $k \geq 2$. Polynomial time algorithms are known for trees, cycles, complete graphs, even with weight [2,4]. Denote by ψ_k the minimum size of a VCP$_k$. Bounds for ψ_k were studied in [2,3,10,11]. Exact algorithms with exponential running time for MVCP$_3$ were studied in [6,12,29]. Parameterized algorithms for MVCP$_3$ were studied in [7,13, 24,28].

In the field of approximation algorithms for VCP$_k$, most studies also focus on the case that $k = 3, 4$. Kardoš et al. [12] presented a randomized approximation algorithm for MVCP$_3$ with an expected performance ratio 23/11. Using local ratio method [21] and primal–dual method [22], respectively, Tu et al presented 2-approximation algorithms for minimum weight VCP$_3$ (MWVCP$_3$). Tu et al. [17,23] also proved that even restricted to cubic graphs, MVCP$_3$ and MVCP$_4$ are still NP-hard, and presented a 1.57-approximation algorithm for MVCP$_3$ and a 2-approximation algorithm for VCP$_4$ on cubic graphs (without weight).

Liu et al. [18] were the first to inject connectivity requirement into the study of MVCP$_k$. A *connected* VCP$_k$ (CVCP$_k$) is a VCP$_k$ which induces a connected subgraph. For fixed integer k, Liu et al. presented a PTAS for the minimum CVCP$_k$ problem on unit disk graph. Also on unit disk graph, Wang et al. [25] obtained a PTAS for the minimum weight CVCP$_3$ problem assuming a c-local condition. Later Wang et al. [26] proposed a weak c-local condition under which they obtained a PTAS for the minimum weight CVCP$_3$ problem on unit ball graph. All these papers used the method of partition and shifting. However, for a ball graph in which balls may have different radii, such a method does not work.

Also keeping in the track of connectivity requirement, Li et al. [16] gave a linear time algorithm for the minimum weight CVCP$_k$ problem on trees, based on which a k-approximation algorithm was proposed for the minimum CVCP$_k$ problem on a general graph with girth (the length of a shortest cycle) at least k.

The minimum weight VCP$_k$ problem is a special case of the *minimum weight vertex deletion problem* [14,15], the goal of which is to find a minimum weight vertex set F such that $G - F$ satisfies a specific property. Using local ratio method, Fujito [9] presented a unified approximation algorithm for such a problem with nontrivial and hereditary graph properties.

In this paper, we use the local search method to give a PTAS for MVCP$_k$ on a ball graph whose heterogeneity is bounded by a constant. Local search method was successfully used to provide PTAS for the minimum hitting set problem for geometric objects which are half spaces in $\mathbb{R}^3$ and admissible regions on the plane [19], the maximum independent set problem of admissible regions on the plane [5], and the minimum dominating set problem on disk graphs [8]. In this paper, we further explore the power of this method by considering the MVCP$_k$ problem.

3. Preliminaries

For a vertex $v \in V(G)$, let $N_G(v)$ be the set of neighbors of v in G. For a subset of vertices $U \subseteq V(G)$, let $N_G(U) = \left(\bigcup_{v\in U} N_G(v)\right) \setminus U$ denote the neighborhood of U.

In [31], Zhang et al. proved the following separator theorem for ball graph.

Theorem 3.1. *Let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be a geometric representation of a d-dimensional ball graph G. Suppose there is a constant K such that every point in $\mathbb{R}^d$ belongs to at most K members of $\mathcal{B}$. Then there exists a constant c depending only on d and K such that for any real number $b > (\frac{2c}{2^{1/d}-1})^d$, the vertex set of G can be partitioned into $A \cup V_1 \cup \cdots \cup V_t$ satisfying the following conditions:*

(i) $t = O(n/b)$;
(ii) *there is no edge between any V_i and V_j for $i \neq j$;*
(iii) $|V_i| \leq b$ *for each* $i = 1, \cdots, t$;
(iv) $|N(V_i) \cap A| \leq b^{1-1/(2d)}$ *for each* $i = 1, \cdots, t$;
(v) $|A| = O(n/\sqrt[2d]{b})$.

A vertex set $U \subseteq V(G)$ is an *independent set* of graph G if no vertices in U are adjacent in G.

Lemma 3.2. *Suppose G is a P_k-free graph. Then $V(G)$ can be partitioned into at most $(k-1)$ independent sets.*

Proof. The lemma can be proved by induction on k. Since a P_2-free graph is an independent set itself, the lemma is true for $k = 2$. Suppose $k \geq 3$ and G is P_k-free. Assume, without loss of generality, that a longest path in G has $k-1$ vertices. Let u_1 be an end of a $(k-1)$-path. If $G - u_1$ still contains a $(k-1)$-path, let u_2 be an end of the $(k-1)$-path. Then u_2 is independent with u_1, since otherwise concatenating edge u_2u_1 with this $(k-1)$-path in $G - u_1$ will yield a k-path in G, contradicting that G is P_k-free. If $G - \{u_1, u_2\}$ still contains a $(k-1)$-path, let u_3 be an end of this $(k-1)$-path. Similarly as the above, $\{u_1, u_2, u_3\}$ is an independent set. Proceeding like this, when the remaining graph no longer contains a $(k-1)$-path, the above u_i's form an independent set of G, denote it as U_1. Now, $G - U_1$ is P_{k-1}-free. By induction hypothesis, $V(G) \setminus U_1$ can be partitioned into at most $(k-2)$ independent sets. Together with U_1, $V(G)$ is partitioned into at most $(k-1)$ independent sets. □

For a ball B in $\mathbb{R}^d$ and a constant α, we use αB to denote the ball with center c_B and radius αr_B. For a set $\mathcal{B}$ of balls in $\mathbb{R}^d$, let $\alpha\mathcal{B} = \{\alpha B : B \in \mathcal{B}\}$.

Lemma 3.3. *Suppose G is a ball graph in $\mathbb{R}^d$ which is P_k-free, $\mathcal{B}$ is a geometric representation of G, and α is a positive constant. Then any point in $\mathbb{R}^d$ is contained in at most $(k-1)(\alpha h(\mathcal{B}) + 1)^d$ balls of $\alpha\mathcal{B}$.*

Proof. By Lemma 3.2, the vertex set of G can be partitioned into $U_1 \cup \ldots \cup U_t$, where $t \leq k-1$ and each U_i is an independent set. We claim that for any $i \in \{1, \ldots, t\}$, any