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A characterization of trees with equal independent domination and secure domination numbers

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ABSTRACT

Let $i(G)$ and $\gamma_s(G)$ be the independent domination number and secure domination number of a graph G, respectively. Merouane and Chellali (2015) [12] proved that $i(T) \le \gamma_s(T)$ for any tree T and asked to characterize the trees T with $i(T) = \gamma_s(T)$. In this paper, we answer the question. We introduce three operations on trees and prove that any tree T with $i(T) = \gamma_s(T)$ can be obtained by these operations.

1. Introduction

All graphs considered in this paper are simple and connected. Let $G = (V, E)$ be a graph. A vertex in G is said to *dominate* itself and every vertex adjacent to it. A set $D \subseteq V$ is said to be a *dominating set* of G if every vertex not in D is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G.

A set $S \subseteq V$ is *independent* if no two vertices in S are adjacent. A dominating set D of G is an *independent dominating set* (IDS) of G if D is independent. The *independent domination number*, denoted by $i(G)$, is the minimum cardinality of an IDS in G. An IDS of G of cardinality $i(G)$ is called an *i-set* of G. A set $D \subseteq V$ is a *double dominating set* of G if every vertex of $V \setminus D$ has at least two neighbors in D and the subgraph induced by D has no isolated vertex. The *double domination number*, denoted by $\gamma_{\times 2}(G)$, is the minimum cardinality of a double dominating set of G.

A dominating set D of a graph G is said to be a *secure dominating set* (SDS) if each vertex $u \in V \setminus D$ is adjacent to a vertex $v \in D$ such that $(D \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The *secure domination number*, denoted by $\gamma_s(G)$, is the minimum cardinality of an SDS of G. An SDS of G of cardinality $\gamma_s(G)$ is called a *γ_s-set* of G.

The problem of secure domination was introduced by Cockayne et al. [8] and has been investigated in the literature [2–7,9–13]. Recently, Merouane and Chellali [12] proved that $i(T) \le \gamma_s(T)$ for any tree T and proposed the following problem.

Problem 1.1. (Merouane and Chellali [12]) Characterize the trees T with $i(T) = \gamma_s(T)$.

In this paper, we give a characterization of the trees T with $i(T) = \gamma_s(T)$.

2. Notations and preliminary results

For a graph $G = (V(G), E(G))$, we denote by $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ the *open neighborhood* of a vertex

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$v \in V(G)$. For a set $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X. An *S-external private neighbor* of a vertex $v \in S$ is a vertex $u \in V(G) \setminus S$ which is adjacent to v but to no other vertex of S. The set of all S-external private neighbors of $v \in S$ is called the *S-external private neighbor set* of v and is denoted $epn(v, S)$. The *degree* of v in G, denoted by $d_G(v)$, is the cardinality of its open neighborhood in G. The *distance* of two vertices u and v in G, denoted by $d_G(u, v)$, is the length of a shortest path between u and v. A vertex of degree one is called a *leaf*. A graph is *trivial* if it has a single vertex. Denote by P_n the path on n vertices.

Let T be a tree. If $u \in V(T)$ is not a leaf of T and $k = \min\{d_T(u, v) : v \in V(T) \text{ and } v \text{ is a leaf of } T\}$, then u is called a *k-stem* of T. A 1-stem is also called a *stem*. A stem v of T is called a *solid stem* if v is adjacent to at least $d_T(v) - 1$ leaves in T. A *pendent edge* of T is an edge incident to a leaf of T. For any edge $uv \in E(T)$, we denote by T_u^v the connected component of $T - uv$ containing the vertex u. Obviously, T_u^v is a subtree of T.

Lemma 2.1. *(Cockayne et al. [8]) A set $D \subseteq V$ is an SDS of a graph G if and only if for each $u \in V \setminus D$, there exists $v \in D$ such that $G[epn(v, D) \cup \{u, v\}]$ is complete.*

Lemma 2.2. *(Merouane and Chellali [12]) For every tree T, $i(T) \le \gamma_s(T)$.*

Note that Lemma 2.2 is obtained from the following two results.

Lemma 2.3. *(Blidia et al. [1]) For every nontrivial tree T, $2i(T) \le \gamma_{\times 2}(T)$, with equality if and only if T has two disjoint i-sets.*

Lemma 2.4. *(Merouane and Chellali [12]) For every connected graph G, $\gamma_{\times 2}(G) \le 2\gamma_s(G)$.*

By Lemmas 2.3 and 2.4, we can immediately obtain the following result.

Corollary 2.5. *Let T be a nontrivial tree. If $i(T) = \gamma_s(T)$, then T has two disjoint i-sets.*

Now we give some properties of the secure domination of graphs, which are useful to characterize the structures of trees.

Lemma 2.6. *Let G be a connected graph with at least three vertices. Then G has a γ_s-set containing all stems of G.*

Proof. Let D be a γ_s-set of G. If G has no stem or D contains all stems of G, then D is a required γ_s-set of G. Otherwise, for any stem x of G such that $x \notin D$, since D is a dominating set of G, we know that each leaf adjacent to x belongs to D. So $(D \setminus \{y\}) \cup \{x\}$ is also a γ_s-set of G, where y is a leaf adjacent to x in T. By repeating this process, we can obtain a γ_s-set of G which contains all stems of G. □

Lemma 2.7. *Let G_1 and G_2 be two subgraphs of a graph G such that $V(G_1) \cap V(G_2) = \emptyset$ and $V(G_1) \cup V(G_2) = V(G)$. If D_i is an SDS of G_i, where $i = 1, 2$, then $D_1 \cup D_2$ is an SDS of G.*

Proof. For any $u \in V(G) \setminus (D_1 \cup D_2)$, we have $u \in V(G_j) \setminus D_j$, where $j = 1$ or 2. Since D_j is an SDS of G_j, by Lemma 2.1, there exists $v \in D_j$ such that the induced subgraph $G'_j = G_j[epn(v, D_j) \cup \{u, v\}]$ is complete. Note that D_{3-j} is an SDS of G_{3-j}, we obtain that $G'_j = G[epn(v, D_1 \cup D_2) \cup \{u, v\}]$. So $D_1 \cup D_2$ is an SDS of G by Lemma 2.1. □

Lemma 2.8. *Let T_x^y be a subtree of T. If D is an SDS of T such that $x \in D$, then the restriction D_x of D to $V(T_x^y)$ is an SDS of T_x^y.*

Proof. For any $u \in V(T_x^y) \setminus D_x$, since D is an SDS of T, by Lemma 2.1, there exists $v \in D$ such that $T[epn(v, D) \cup \{u, v\}]$ is complete. Note that such vertex v must in $V(T_x^y)$, so $T_x^y[epn(v, D_x) \cup \{u, v\}]$ is also complete. Therefore, D_x is an SDS of T_x^y. □

3. A characterization of the trees T with $i(T) = \gamma_s(T)$

Let T be a tree. If $|V(T)| \le 2$, then $i(T) = \gamma_s(T)$. Thus, in what follows, we only consider the case of $|V(T)| \ge 3$. We first need to prove the following useful result.

Lemma 3.1. *Let T be a tree such that $i(T) = \gamma_s(T)$. Then any stem of T is adjacent to exactly one leaf.*

Proof. Suppose that there exists a stem x of T which is adjacent to at least two leaves y and z. By Lemma 2.6, T has a γ_s-set containing all stems of T, denoted by D. Then at least one of y and z, say y, belongs to D. Let $T' = T - y$ and $D' = D \setminus \{y\}$. Since $x \in D$, by Lemma 2.8 we obtain that D' is an SDS of T'. So $\gamma_s(T') \le |D| - 1$. On the other hand, for any γ_s-set D'_0 of T', $D'_0 \cup \{y\}$ is an SDS of T by Lemma 2.7. Then $\gamma_s(T) \le |D'_0| + 1 = \gamma_s(T') + 1$. So $\gamma_s(T') = |D| - 1 = \gamma_s(T) - 1$.

Now we prove that $i(T') = i(T) - 1$. Let D'_1 be an i-set of T'. If $x \in D'_1$, then D'_1 is an IDS of T; otherwise, $D'_1 \cup \{y\}$ is an IDS of T. So $i(T) \le |D'_1| + 1 = i(T') + 1$. Note that $i(T) = \gamma_s(T)$ and $i(T') \le \gamma_s(T')$ by Lemma 2.2, we have $i(T') \le i(T) - 1$. So $i(T') = i(T) - 1$.

Therefore, $\gamma_s(T') = \gamma_s(T) - 1 = i(T) - 1 = i(T')$. It follow from Corollary 2.5 that T' has two disjoint i-sets. Since for any IDS of T', the vertex z is dominated only by x or z, so there exists an i-set of T' containing x, denoted by D'_1. We can see that D'_1 is also an IDS of T. Thus, $i(T) \le |D'_1| = i(T')$, a contradiction. □

As a straightforward consequence of Lemma 3.1, we have:

Corollary 3.2. *Let T be a tree and x be a solid stem of T. If $i(T) = \gamma_s(T)$, then $d_T(x) = 2$.*

Lemma 3.3. *Let T be a tree such that $i(T) = \gamma_s(T)$, x be a solid stem and y be the unique leaf adjacent to x in T. For any γ_s-set D of T, if $x \in D$, then $y \notin D$.*