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On the two largest distance eigenvalues of graph powers

Rundan Xing [a], Bo Zhou [b,*]

[a] School of Computer Science, Wuyi University, Jiangmen 529020, PR China
[b] School of Mathematical Sciences, South China Normal University, Guangzhou 510631, PR China

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ABSTRACT

We give sharp lower bounds for the largest and the second largest distance eigenvalues of the k-th power of a connected graph, determine all trees and unicyclic graphs for which the second largest distance eigenvalues of the squares are less than $\frac{\sqrt{5}-3}{2}$, and determine the unique n-vertex trees of which the squares achieve minimum and second-minimum largest distance eigenvalues, as well as the unique n-vertex trees of which the squares achieve minimum, second-minimum and third-minimum second largest distance eigenvalues.

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1. Introduction

We consider simple undirected graphs. Let G be a connected graph. The distance between two vertices of G is the length of a shortest path connecting them in G. For a positive integer k, the k-th power of G, denoted by G^k, is the (simple) graph obtained from G by adding an edge between each pair of vertices with distance at most k [6, p. 82]. Obviously, if G is of diameter d, then $G^1 \cong G$ and G^i is isomorphic to the complete graph for integer $i \geq d$. In particular, the graph G^2 is also called the square of G. Graph powers are useful in designing efficient algorithms for certain combinatorial optimization problems, see, e.g., [2,5]. In distributed computing, the k-th power of graph G represents the possible flow of information during k rounds of communication in a distributed network of processors organized according to G [17]. For various aspects of graph powers, see, e.g., [1,7,9,13,15,16,20].

Let G be a connected graph with vertex set $V(G) = \{v_1, \dots, v_n\}$. For $1 \leq i, j \leq n$, let $d_G(v_i, v_j)$ be the distance between vertices v_i and v_j in G. The distance matrix of G is the $n \times n$ matrix $D(G) = [d_G(v_i, v_j)]$. The distance eigenvalues of G, denoted by $\lambda_1(G), \dots, \lambda_n(G)$, are the eigenvalues of $D(G)$, arranged in non-increasing order. We call $\lambda_k(G)$ the k-th largest distance eigenvalue of G. The study of distance eigenvalues dates back to the classical work of Graham and Pollack [12], Edelberg et al. [10] and Graham and Lovász [11] in 1970s. A relationship was established in [12] between the number of negative distance eigenvalues and the addressing problem in data communication systems. Until now the distance eigenvalues have been studied extensively, see, e.g. [3,19]. For a tree G, Graham and Pollack [12] showed that $\lambda_2(G) < 0$. For a unicyclic graph G, Bapat et al. [4] showed that $\lambda_2(G) < 0$ if its girth is odd and $\lambda_2(G) = 0$ if its girth is even. In [21], we characterized all connected graphs with second largest distance eigenvalue less than $-2 + \sqrt{2}$, and all trees with second largest distance eigenvalue less than $-\frac{1}{2}$. Thus, it is of interest to study the distance eigenvalues of some particular classes of graphs. Now we consider the graph powers, of which the adjacency and Laplacian eigenvalues have been studied, see [20,9]. The bounds of a graph invariant are important information for the graph in the sense that they establish the range of the graph invariant.

* Corresponding author.
E-mail addresses: rundanxing@126.com (R. Xing), zhoubo@scnu.edu.cn (B. Zhou).

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In this paper, we give sharp lower bounds for the (first) largest and the second largest distance eigenvalues of the k-th power of a connected graph, and determine all trees and unicyclic graphs G such that $\lambda_2(G^2) < \frac{\sqrt{5}-3}{2}$. We also determine the unique n-vertex trees, the squares of which achieve minimum and second-minimum largest distance eigenvalues, as well as the unique n-vertex trees, the squares of which achieve minimum, second-minimum and third-minimum second largest distance eigenvalues.

2. Preliminaries and lemmas

In this section, we give preliminaries and some lemmas that will be used later.

Let G be a connected graph with two nonadjacent vertices $u, v \in V(G)$. Let $G' = G + uv$ be the graph obtained from G by adding the edge uv. Note that $D(G)$ is irreducible, $D(G')$ is nonnegative, and each entry of $D(G')$ does not exceed the corresponding one of $D(G)$ with $d_{G'}(u, v) < d_G(u, v)$. The following result follows, see [18, p. 38].

Lemma 2.1. *Let G be a connected graph with two nonadjacent vertices $u, v \in V(G)$. Then $\lambda_1(G) > \lambda_1(G + uv)$.*

Let G be a connected graph, and H an induced subgraph of G. If H is connected and $d_H(u, v) = d_G(u, v)$ for all $\{u, v\} \subseteq V(H)$, then write $H \trianglelefteq G$. By the interlacing theorem [14, pp. 185–186], we have

Lemma 2.2. *Let G be a connected graph, and H a non-trivial induced subgraph of G with $H \trianglelefteq G$. Then $\lambda_2(G) \geq \lambda_2(H)$.*

Let G be a graph. For $u \in V(G)$, $N_G(u)$ denotes the set of neighbors of vertex u in G, and $d_G(u) = |N_G(u)|$ denotes the degree of vertex u in G.

Let $diam(G)$ be the diameter of a connected graph G.

Let K_n, P_n, S_n and C_n be respectively the complete graph, the path, the star and the cycle on n vertices.

Let I_n be the $n \times n$ identity matrix, and $J_{m\times n}$ the $m \times n$ all-one matrix.

For positive integers r, s and t, let $A = A(r, s, t)$ be the following symmetric block matrix

$$\begin{bmatrix} J_{r\times r} & J_{r\times s} & 2J_{r\times t} \\ J_{s\times r} & J_{s\times s} & J_{s\times t} \\ 2J_{t\times r} & J_{t\times s} & J_{t\times t} \end{bmatrix},$$

whose rows and columns are partitioned according to the partition $\{1, \ldots, r + s + t\} = \{1, \ldots, r\} \cup \{r + 1, \ldots, r + s\} \cup \{r + s + 1, \ldots, r + s + t\}$. Obviously, A is of rank 3, and thus 0 is one of its eigenvalues with multiplicity at least $r + s + t - 3$. The quotient matrix of A is the matrix $B = B(r, s, t)$ whose entries are the average row sums of the corresponding blocks of A, i.e.,

$$B = \begin{bmatrix} r & s & 2t \\ r & s & t \\ 2r & s & t \end{bmatrix}.$$

Since each block of A has constant row sum, the above partition is equitable (or regular). By [8, Lemma 2.3.1], all eigenvalues of B are also eigenvalues of A. Let $f_{r,s,t}(\lambda) = \det(\lambda I_3 - B)$. By direct calculation, we have

$$f_{r,s,t}(\lambda) = \lambda^3 - (r + s + t)\lambda^2 - 3rt\lambda + rst.$$

Since $f_{r,s,t}(0) = rst \neq 0$, the eigenvalues of B are non-zero. Thus the eigenvalues of A are 0 with multiplicity $r + s + t - 3$, and the three roots of the equation $f_{r,s,t}(\lambda) = 0$.

Let $D_{n,p}$ be the n-vertex double star obtained by adding an edge between the centers of the stars of S_{p+1} and S_{n-p-1}, where $1 \leq p \leq \lfloor \frac{n-2}{2} \rfloor$.

Lemma 2.3. *For $2 \leq p \leq \lfloor \frac{n-2}{2} \rfloor$ and $i = 1, 2$, we have $\lambda_i(D^2_{n,p}) > \lambda_i(D^2_{n,p-1})$.*

Proof. Let $G = D_{n,p}$. Label by $v_1, \ldots, v_n$ the vertices of G, where $d_G(v_1) = \cdots = d_G(v_p) = 1$, $d_G(v_{p+1}) = p + 1$, $d_G(v_{p+2}) = n - 1 - p$, $d_G(v_{p+3}) = \cdots = d_G(v_n) = 1$, $N_G(v_{p+1}) = \{v_1, \ldots, v_p, v_{p+2}\}$ and $N_G(v_{p+2}) = \{v_{p+1}, v_{p+3}, \ldots, v_n\}$. Then, with respect to the partition $V(G^2) = V(G) = \{v_1, \ldots, v_p\} \cup \{v_{p+1}, v_{p+2}\} \cup \{v_{p+3}, \ldots, v_n\}$, we have

$$D(G^2) + I_n = A(p, 2, n - 2 - p).$$

Thus the eigenvalues of $D(G^2) + I_n$, which are $\lambda_1(G^2) + 1, \ldots, \lambda_n(G^2) + 1$, are given by 0 with multiplicity $n - 3$, and the three roots of the equation $f_{p,2,n-2-p}(\lambda) = 0$. Since $diam(G) = 3$, we have $P_4 \trianglelefteq G$, and thus $P_4^2 \trianglelefteq G^2$. By Lemma 2.2, $\lambda_1(G^2) + 1 \geq \lambda_2(G^2) + 1 \geq \lambda_2(P_4^2) + 1 = \frac{3-\sqrt{17}}{2} + 1 > 0$, implying that $\lambda_1(G^2) + 1$ and $\lambda_2(G^2) + 1$ are respectively the largest and the second largest roots of $f_{p,2,n-2-p}(\lambda) = 0$, i.e., $\lambda_1(G^2)$ and $\lambda_2(G^2)$ are respectively the largest and the second largest roots of $g_p(\lambda) = 0$, where

$$\begin{aligned} g_p(\lambda) &= f_{p,2,n-2-p}(\lambda + 1) \\ &= \lambda^3 - (n - 3)\lambda^2 - (3pn + 2n - 3p^2 - 6p - 3)\lambda \\ &\quad - pn - n + p^2 + 2p + 1. \end{aligned}$$

Let $G' = D_{n,p-1}$. Then $\lambda_1(G'^2)$ and $\lambda_2(G'^2)$ are respectively the largest and the second largest roots of $g_{p-1}(\lambda) = 0$. By direct calculation, we have

$$g_p(\lambda) - g_{p-1}(\lambda) = -(n - 2p - 1)(3\lambda + 1).$$

Thus $g_p(\lambda_1(G'^2)) = g_p(\lambda_1(G'^2)) - g_{p-1}(\lambda_1(G'^2)) = -(n - 2p - 1)(3\lambda_1(G'^2) + 1) < 0$ by noting that $\lambda_1(G'^2) > 0$, which, together with the fact that $g_p(\lambda) \geq 0$ for $\lambda \geq \lambda_1(G^2)$, implies that $\lambda_1(G^2) > \lambda_1(G'^2)$.

Note that $2 \leq p \leq \lfloor \frac{n-2}{2} \rfloor \leq \frac{n-2}{2}$, which implies that $n \geq 2p + 2$. By direct check, $g_{p-1}(-\frac{1}{3}) = -\frac{4}{9}n + \frac{8}{27} < 0$ and $g_{p-1}(-1) = 2n(p - 1) - 2p^2 + 2 \geq 2p^2 - 2 > 0$, implying that $\lambda_2(G'^2) \in (-1, -\frac{1}{3})$. Then $g_p(\lambda_2(G'^2)) = g_p(\lambda_2(G'^2)) - g_{p-1}(\lambda_2(G'^2)) = -(n - 2p - 1)(3\lambda_2(G'^2) + 1) > 0$, which, together with the fact that $\lambda_1(G^2) > \lambda_1(G'^2)$, implies that $\lambda_2(G^2) > \lambda_2(G'^2)$. □

Let G_1 and G_2 be two vertex-disjoint graphs with $u \in V(G_1)$ and $v \in V(G_2)$. The coalescence $G_1(u) \circ G_2(v)$ is

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